

Zero-sum stopping games with asymmetric information.

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January 27, 2016

Abstract

We study a model of two-player, zero-sum, stopping games with asymmetric information. We assume that the payoff depends on two continuous-time Markov chains $(X_t), (Y_t)$, where (X_t) is only observed by player 1 and (Y_t) only by player 2, implying that the players have access to stopping times with respect to different filtrations. We show the existence of a value in mixed stopping times and provide a variational characterization for the value as a function of the initial distribution of the Markov chains. We also prove a verification theorem for optimal stopping rules which allows to construct optimal stopping times. Finally we use our results to solve explicitly two generic examples.

Acknowledgments : The two authors gratefully acknowledge the support of the Agence Nationale de la Recherche, under grant ANR JEUDY, ANR-10-BLAN 0112.

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1 Introduction

In this paper we consider a two player zero-sum stopping game with asymmetric information. The payoff depends on two independent continuous time Markov chains $(X_t, Y_t)_{t \geq 0}$ with finite state space $K \times L$, commonly known initial law $p \otimes q$ and infinitesimal generators $R = (R_{k,k'})_{k,k' \in K}$ and $Q = (Q_{\ell,\ell'})_{\ell,\ell' \in L}$. We assume that X is only observed by player 1 while Y is only observed by player 2. The fact that the game was not stopped up to some time gives to each player some additional information about the unknown state. This is a crucial point as it implies that players have to take into account which information they generate about their private state when searching for optimal strategies. In consequence, our analysis is significantly different to that of classical stopping games.

We prove the existence of the value $V(p, q)$ in mixed stopping times allowing the players to randomize their stopping decision. Mixed stopping times have already been studied by Baxter and Chacon [3] and Meyer [25] in a continuous time setting and applied to stopping games by Vieille and Touzi [33] and Laraki and Solan [23]. We also refer to the recent work of Shmaya and Solan [32] for a concise study of this type of stopping times. We work under the common assumptions on the payoffs used by Lepeltier and Maingueneau [24], in order to provide a variational characterization for V . Moreover, we show how variational characterization can be applied to determine optimal strategies in the case of incomplete information on both sides. This result is new, since up to now similar characterizations are only known for the case of incomplete information on one side, and applies to a wide class of examples.

The variational characterization for V that we provide can be seen as an extension (in a simple case) of the classical semi-harmonic characterization for stopping games of Markov processes with symmetric information (see e.g. Friedman [14], Eckström and Peskir [12]) to models with asymmetric information. It is reminiscent of the variational representation for the value of repeated games with asymmetric information given by Rosenberg and Sorin [29] and Laraki [22]. It is also equivalent to a first-order PDE with convexity constraints as introduced by Cardaliaguet in [4]. Using appropriate PDEs with convexity constraints, the mentioned results were later extended to particular classes of continuous-time games by Cardaliaguet and Rainer [5, 7], to continuous time stopping games by Grün [19] and to continuous time limit of repeated games by Gensbittel [15, 16].

Most of the literature on dynamic games with asymmetric information deals with models where the payoff-relevant parameters of the game that are partially unknown (say information parameters) do not evolve over time. Some recent works focus on models of dynamic games with asymmetric information and evolving information parameters (see e.g. Renault [28], Neyman [26], Gensbittel and Renault [18], Cardaliaguet et al. [8], Gensbittel [17]). All these works consider dynamic games with current or terminal payoffs while in the present work we study the case of continuous-time stopping games with time-evolving information parameters.

The paper is structured as follows. First we give a description of the model and the main definitions. In the third section we establish existence and uniqueness of the value by a variational characterization using PDE methods. We use this result in the following section to characterize optimal strategies for both players. In section 5 we present two examples where explicit expressions for the value as well as optimal strategies for both players are provided. The appendix collects auxiliary results and some technical proofs.

2 Model

2.1 Notations

For any topological space E , $\mathcal{B}(E)$ denotes its Borel σ -algebra, $\Delta(E)$ denotes the set of Borel probability distributions on E and δ_x denotes the Dirac measure at $x \in E$. Finite sets are endowed with the discrete topology and Cartesian products with the product topology. If E is finite, then $|E|$ denotes its cardinal and $\Delta(E)$ is identified with the unit simplex of \mathbb{R}^E . $\langle \cdot, \cdot \rangle$ and $|\cdot|$ applied to vectors stand the usual scalar product and Euclidean norm while ${}^T M$ stands for the transpose of a matrix M .

2.2 The dynamics

Let K, L be two non-empty finite sets. We consider two independent continuous-time, homogeneous Markov chains $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ with state space K and L , initial laws $p \in \Delta(K)$, $q \in \Delta(L)$ and infinitesimal generators $R = (R_{k,k'})_{k,k' \in K}$ and $Q = (Q_{\ell,\ell'})_{\ell,\ell' \in L}$ respectively. $R_{k,k'}$ represents as usual the jump intensity of the process X from state k to state k' when $k' \neq k$ and $R_{k,k} = -\sum_{k' \neq k} R_{k,k'}$.

We denote \mathbb{P}_p^X the law of the process X defined on the canonical space of K -valued càdlàg trajectories $\Omega_X = \mathbb{D}([0, \infty), K)$ and \mathbb{P}_q^Y the law of the process Y defined on the space $\Omega_Y = \mathbb{D}([0, \infty), L)$. Furthermore, let us define

$$(\Omega, \mathcal{A}; \mathbb{P}_{p,q}) := (\Omega_X \times \Omega_Y, \mathcal{F}_\infty^X \otimes \mathcal{F}_\infty^Y, \mathbb{P}_p^X \otimes \mathbb{P}_q^Y).$$

We will identify the right-continuous filtrations \mathcal{F}^X and \mathcal{F}^Y (see Theorem 26 p. 304 in [2]) as filtrations defined on Ω as well as \mathcal{F}_∞^X -measurable random variables on Ω as \mathcal{F}_∞^X -measurable variables defined on Ω_X (and similarly for Y).

We consider a zero-sum stopping game, where Player 1 observes the trajectory of X , while Player 2 observes the trajectory of Y . So according to their information the dynamics of the game for Player 1 are basically given by

$$(\mathbb{P}_{p,q}[X_t = k | \mathcal{F}_s^X])_{k \in K} = e^{(t-s) {}^T R} \delta_{X_s}, \quad (\mathbb{P}_{p,q}[Y_t = \ell | \mathcal{F}_\infty^X])_{\ell \in L} = e^{t {}^T Q} q,$$

while for Player 2 they are given by

$$(\mathbb{P}_{p,q}[X_t = k | \mathcal{F}_\infty^Y])_{k \in K} = e^{t {}^T R} p, \quad (\mathbb{P}_{p,q}[Y_t = \ell | \mathcal{F}_s^Y])_{\ell \in L} = e^{(t-s) {}^T Q} \delta_{Y_s},$$

where we use the independence of X and Y .

2.3 Mixed stopping times and payoff function

Let \mathcal{T}^X denote the set of \mathcal{F}^X stopping times and \mathcal{T}^Y denote the set of \mathcal{F}^Y stopping times.

Definition 2.1. A mixed stopping time of the filtration \mathcal{F}^X on Ω is an $\mathcal{F}_\infty^X \otimes \mathcal{B}([0, 1])$ -measurable map μ defined on $\Omega \times [0, 1]$ such that $\mu(\cdot, u) \in \mathcal{T}^X$ for all $u \in [0, 1]$. We denote \mathcal{T}_r^X the set of mixed \mathcal{F}^X -stopping times.

The definition mixed stopping times of the filtration \mathcal{F}^Y is similar. We denote \mathcal{T}_r^Y the set of mixed \mathcal{F}^Y -stopping times.

A random time is a $\mathcal{B}([0, 1])$ -measurable map $\mu : [0, 1] \rightarrow [0, +\infty]$. The set of random times is denoted \mathcal{T}_r^\emptyset .

Let $r > 0$ denote a fix discount rate and $f \geq h$ two real-valued functions defined on $K \times L$. The players choose mixed stopping times $\mu(\omega, u) \in \mathcal{T}_r^X$ and $\nu(\omega, v) \in \mathcal{T}_r^Y$ respectively, in order to maximize (resp. minimize) the expected payoff:

$$\mathbb{E}_{p,q} \left[\int_0^1 \int_0^1 J(\mu, \nu)(\omega, u, v) dudv \right], \quad (2.1)$$

where

$$J(\mu, \nu)(\omega, u, v) := e^{-r\nu} f(X_\nu, Y_\nu) \mathbb{1}_{\nu < \mu} + e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \nu}. \quad (2.2)$$

Furthermore we set

$$\bar{J}(\mu, \nu) := \int_0^1 \int_0^1 J(\mu, \nu)(\omega, u, v) dudv.$$

The upper value of the game is defined by

$$V^+(p, q) := \inf_{\nu \in \mathcal{T}_r^Y} \sup_{\mu \in \mathcal{T}_r^X} \mathbb{E}_{p,q} [\bar{J}(\mu, \nu)], \quad (2.3)$$

the lower value by

$$V^-(p, q) := \sup_{\mu \in \mathcal{T}_r^X} \inf_{\nu \in \mathcal{T}_r^Y} \mathbb{E}_{p,q} [\bar{J}(\mu, \nu)], \quad (2.4)$$

where by definition $V^-(p, q) \leq V^+(p, q)$. When there is equality, we say that the game has a value $V := V^- = V^+$.

Let us comment the notion of mixed (or randomized) stopping times. Our definition corresponds to the classical notion of mixed strategy in a game: for each player, a mixed strategy is a probability distribution over his possible pure (i.e. standard) strategies. Following Aumann [1], a natural way to define mixed strategies when the set of pure strategies is a set of measurable maps but has no simple measurable structure (as \mathcal{T}^X and \mathcal{T}^Y in our model), is to introduce an auxiliary probability space which is used as a randomization device by the player, allowing him to choose at random a pure strategy. With such a definition, one has only to require the mixed strategy to be jointly measurable for the expected payoff to be well-defined. We refer the reader to [32] for a discussion of the different equivalent definitions of randomized stopping times. From an analytic point of view, the set of randomized stopping times \mathcal{F}_r^X is the closed convex hull for the weak topology (which may coincide with the closure in some cases but not in our model¹) of the set of stopping times $\tau \in \mathcal{T}^X$ seen as linear forms $\mathbb{E}[Z_\tau]$ acting on the class of bounded \mathcal{F}_∞^X -measurable continuous processes Z , and it was proved to be compact in [3] and [25]. We do not use such topological properties of mixed stopping times and our existence result is not based on some general minmax theorem, but convexity plays a central role in our analysis, as can be seen in the statement of Theorem 3.3.

It is well known that in games where both players have the same information the value exists under fairly general assumptions, when the players are only allowed to choose stopping times adapted to their common information. The existence of the value implies that there is no loss for either of the players if they give their adversary the advantage of playing second.

¹Indeed, if $X_0 = 0$, for sufficiently small $\alpha > 0$, it is not possible to obtain the linear form $\frac{1}{2}(\mathbb{E}_{\delta_0}[Z_0 + Z_\alpha])$ as a limit of classical stopping times. This is due to the fact that the event $\{\forall s \in [0, \alpha], X_s = 0\}$ has probability close to 1 for small α , implying that no stopping time in \mathcal{T}^X can be equal to 0 and α with probabilities close to 1/2.

For example in $V^+(p, q)$ the maximizer has an information advantage by knowing exactly which strategies he is facing. In games with incomplete information, existence of a value in non-randomized strategies is in general not true, as shown in the first example in section 5. The intuitive explanation is that the first player would, by choosing non randomized strategies, reveal too much information to the second player about the process X , which is not observed by Player 2. In contrast, using randomized strategies allow each player to manipulate the beliefs of his opponent. This translates into the fact that the value is concave in p and convex in q (see Lemma 3.5) and, as can be seen in the examples of section 5, in general non-linear with respect to the initial distributions $(p, q) \in \Delta(K) \times \Delta(L)$.

3 Existence and characterization of the value

3.1 Result

Our first result is the existence of the value together with a variational characterization, which is a first-order PDE with convexity constraints. These constraints are expressed using the notion of extreme points as in [7, 22, 29].

Definition 3.1. A function $g : \Delta(K) \times \Delta(L) \rightarrow \mathbb{R}$ is said to be a saddle function if it is concave with respect to $p \in \Delta(K)$ and convex with respect to $q \in \Delta(L)$.

For any $q \in \Delta(L)$, the set of extreme points $\text{Ext}(g(\cdot, q))$ is defined as the set of all $p \in \Delta(K)$ such that

$$(p, g(p, q)) = \lambda(p_1, g(p_1, q)) + (1 - \lambda)(p_2, g(p_2, q))$$

with $\lambda \in (0, 1)$ and $p_1, p_2 \in \Delta(K)$ implies $p_1 = p_2 = p$. The set of extreme points $\text{Ext}(g(p, \cdot))$ for any $p \in \Delta(K)$ is defined in a similar way.

Remark 3.2. One can check easily that $\text{Ext}(g(\cdot, q))$ is the set of $p \in \Delta(K)$ such that $(p, g(p, q))$ is an extreme point (in the usual sense, see Definition 3.11) of the hypograph of $g(\cdot, q)$ defined by $\{(p', t) \in \Delta(K) \times \mathbb{R} \mid t \leq g(p', q)\}$. That $p \in \text{Ext}(g(\cdot, q))$ means that $g(\cdot, q)$ is strictly concave at p , i.e. not affine in any non-trivial segment containing p . Note also that this definition implies that all the Dirac masses δ_k for $k \in K$ always belong to $\text{Ext}(g(\cdot, q))$. Similarly properties hold for $\text{Ext}(g(p, \cdot))$.

Let f, h be extended linearly on the set $\Delta(K)$, i.e.

$$\forall (p, q) \in \Delta(K) \times \Delta(L), f(p, q) := \sum_{(k, \ell) \in K \times L} p_k q_\ell f(k, \ell), \quad h(p, q) := \sum_{(k, \ell) \in K \times L} p_k q_\ell h(k, \ell).$$

Theorem 3.3. For all $(p, q) \in \Delta(K) \times \Delta(L)$, the game has a value

$$V(p, q) := V^+(p, q) = V^-(p, q),$$

and V is the unique Lipschitz saddle function on $\Delta(K) \times \Delta(L)$ such that:

(Subsolution) $\forall q \in \Delta(L), \forall p \in \text{Ext}(V(\cdot, q)),$

$$\begin{aligned} \max\{\min\{rV(p, q) - \vec{D}_1 V(p, q; {}^\top R p) - \vec{D}_2 V(p, q; {}^\top Q q); V(p, q) - h(p, q)\}; \\ V(p, q) - f(p, q)\} \leq 0, \end{aligned} \quad (3.1)$$

(Supersolution) $\forall p \in \Delta(K), \forall q \in \text{Ext}(V(p, \cdot)),$

$$\max\{\min\{rV(p, q) - \vec{D}_1 V(p, q; {}^\top R p) - \vec{D}_2 V(p, q; {}^\top Q q); V(p, q) - h(p, q)\}; \\ V(p, q) - f(p, q)\} \geq 0, \quad (3.2)$$

where $\vec{D}_1 V(p, q; \xi)$ and $\vec{D}_2 V(p, q; \zeta)$ denote the directional derivatives of V at (p, q) with respect to the first and second variables in the directions ξ and ζ .

Let us comment (3.1) and (3.2). If p is an extreme point, i.e. $p \in \text{Ext}(V(\cdot, q))$, (3.1) is a standard subsolution property for the obstacle PDE,

$$\max\{\min\{rV(p, q) - \vec{D}_1 V(p, q; {}^\top R p) - \vec{D}_2 V(p, q; {}^\top Q q); V(p, q) - h(p, q)\}; \\ V(p, q) - f(p, q)\} = 0, \quad (3.3)$$

while if q is an extreme point, i.e. $q \in \text{Ext}(V(p, \cdot))$, (3.2) is the supersolution property for this PDE. We note that because V is a saddle function, it is reasonable to state the equation in the strong form since we can define directional derivatives. This allows us also to derive the comparison principle in section 3.5 with classical tools.

Moreover the game where both players do not observe (X_t, Y_t) corresponds to a game with deterministic dynamic on $\Delta(K) \times \Delta(L)$ given by the marginal distribution of (X_t, Y_t) , i.e. $(e^{t^\top R} p, e^{t^\top Q} q)$ for some initial values $(p, q) \in \Delta(K) \times \Delta(L)$. It is well known that the value of is characterized as the unique viscosity solution to

$$\max\{\min\{rw(p, q) - \langle \nabla_p w(p, q), {}^\top R p \rangle - \langle \nabla_q w(p, q), {}^\top Q p \rangle; w(p, q) - h(p, q)\}; \\ w(p, q) - f(p, q)\} = 0, \quad (3.4)$$

which is (3.3) with a weaker a priori regularity. So loosely speaking, for extreme points of the value function of the game with incomplete information, $V(p, q)$ solves the same variational inequalities as the value of the game where both players do not observe (X_t, Y_t) . However, the set of extreme points might be very small (but it contains at least all the Dirac masses) and our characterization also requires V to be a saddle function. It follows that the value function $V(p, q)$ significantly differs from the solution to (3.4) in general, as seen in the second example in section 5 where the set of extreme points of V contains only one point which is not a Dirac mass (see Remark 5.4).

3.2 Properties of V^+, V^- and their concave, convex conjugates

First we note the following fact.

Remark 3.4. We can replace the infimum over \mathcal{T}_r^Y by an infimum over \mathcal{T}^Y in the definition of V^- (and the symmetric property for V^+) i.e.

$$V^-(p, q) = \sup_{\mu \in \mathcal{T}_r^X} \inf_{\nu \in \mathcal{T}^Y} \bar{J}(\mu, \nu), \\ V^+(p, q) = \inf_{\nu \in \mathcal{T}_r^Y} \sup_{\mu \in \mathcal{T}^X} \bar{J}(\mu, \nu). \quad (3.5)$$

Indeed, using Fubini theorem, for any $\mu \in \mathcal{T}_r^X$, we have

$$\inf_{\nu \in \mathcal{T}_r^Y} \mathbb{E}_{p, q}[\bar{J}(\mu, \nu)] = \inf_{\nu \in \mathcal{T}_r^Y} \int_0^1 \mathbb{E}_{p, q}[J(\mu, \nu) du] dv \geq \inf_{\nu \in \mathcal{T}_r^Y} \inf_{v \in [0, 1]} \mathbb{E}_{p, q}[J(\mu, \nu(\cdot, v)) du],$$

and since for all $\nu \in \mathcal{T}_r^Y$ and all $v \in [0, 1]$, $\nu(\cdot, v)$ is a stopping time in \mathcal{T}^Y , we deduce that

$$\inf_{\nu \in \mathcal{T}_r^Y} \bar{J}(\mu, \nu) \geq \inf_{\nu \in \mathcal{T}^Y} \bar{J}(\mu, \nu),$$

which proves the result since $\mathcal{T}^Y \subset \mathcal{T}_r^Y$.

We summarize the properties of V^+, V^- in the following lemma.

Lemma 3.5. V^+ and V^- are Lipschitz saddle functions and

$$\forall p, q \in \Delta(K) \times \Delta(L), h(p, q) \leq V^-(p, q) \leq V^+(p, q) \leq f(p, q). \quad (3.6)$$

Proof. We prove the claim only for V^+ since the proof V^- is similar. It is easily seen that inequality (3.6) follows immediately from the definitions. We have $V^- \leq V^+$ and V^- can be estimated from below by setting $\mu = 0$ while V^+ can be estimated from above by setting $\nu = 0$ taking into consideration that the obstacles satisfy $h(p, q) \leq f(p, q)$ for all $p, q \in \Delta(K) \times \Delta(L)$.

Furthermore we note that for any $\mu \in \mathcal{T}_r^X, \nu \in \mathcal{T}_r^Y$, it holds by conditioning

$$\begin{aligned} \mathbb{E}_{p,q} [\bar{J}(\mu, \nu)] &= \sum_{k \in K, \ell \in L} \mathbb{P}[X_0 = k] \mathbb{P}[Y_0 = \ell] \mathbb{E}_{p,q} [\bar{J}(\mu, \nu) | X_0 = k, Y_0 = \ell] \\ &= \sum_{k \in K, \ell \in L} p_k q_\ell \mathbb{E}_{\delta_k, \delta_\ell} [\bar{J}(\mu, \nu)], \end{aligned} \quad (3.7)$$

δ_k, δ_ℓ denoting the Dirac masses at k, ℓ identified with the k -th, ℓ -th vectors in the canonical bases of \mathbb{R}^K and \mathbb{R}^L respectively.

In order to show the Lipschitz continuity, let $p, p' \in \Delta(K), q, q' \in \Delta(L)$ such that $0 < V^+(p, q) - V^+(p', q')$. Choosing $\nu^* \in \mathcal{T}_r^Y$ ε -optimal for $V^+(p', q')$ and $\mu^* \in \mathcal{T}_r^X$ ε -optimal for $\sup_{\mu \in \mathcal{T}_r^X} \mathbb{E}_{p,q} [\bar{J}(\mu, \nu^*)]$ we have

$$0 < V^+(p, q) - V^+(p', q') \leq \mathbb{E}_{p,q} [\bar{J}(\mu^*, \nu^*)] - \mathbb{E}_{p',q'} [\bar{J}(\mu^*, \nu^*)] + 2\varepsilon \quad (3.8)$$

for ε arbitrarily small. The Lipschitz continuity follows then immediately by (3.7).

Furthermore we claim that:

$$\begin{aligned} V^+(p, q) &= \inf_{\nu \in \mathcal{T}_r^Y} \sup_{\mu \in \mathcal{T}^X} \mathbb{E}_{p,q} [\bar{J}(\mu, \nu)] \\ &= \inf_{\nu \in \mathcal{T}_r^Y} \sum_{k \in K} p_k \left(\sup_{\mu \in \mathcal{T}^X} \mathbb{E}_{\delta_k, q} [\bar{J}(\mu, \nu)] \right). \end{aligned} \quad (3.9)$$

Indeed, $V^+(p, q)$ is clearly less or equal than the second line in the above equation. To prove the reverse inequality, for any $\nu \in \mathcal{T}_r^Y$, and any $k \in K$, let μ^k be some ε -optimal stopping time for the problem $\sup_{\mu \in \mathcal{T}^X} \mathbb{E}_{\delta_k, q} [\bar{J}(\mu, \nu)]$. Setting $\mu = \sum_{k \in K} \mathbb{1}_{X_0=k} \mu^k$ we note that

$$\sum_{k \in K} p_k \mathbb{E}_{\delta_k, q} [\bar{J}(\mu^k, \nu)] = \mathbb{E}_{p,q} [\bar{J}(\mu, \nu)]$$

and (3.9) follows by sending ε to zero.

We deduce from (3.9) that $p \rightarrow V^+(p, q)$ is concave as an infimum of affine functions. The convexity in q follows by the classical splitting method. Let $q_1, q_2, q \in \Delta(L)$, $\lambda \in (0, 1)$ such that

$$q = \lambda q_1 + (1 - \lambda) q_2.$$

We choose $\nu_1 \in \mathcal{T}_r^Y$, $\nu_2 \in \mathcal{T}_r^Y$ ε -optimal for $V^+(p, q_1)$ and $V^+(p, q_2)$ respectively. Then we will construct $\nu \in \mathcal{T}_r^Y$ such that

$$\mathbb{E}_{p,q} [\bar{J}(\mu, \nu)] = \lambda \mathbb{E}_{p,q_1} [\bar{J}(\mu, \nu_1)] + (1 - \lambda) \mathbb{E}_{p,q_2} [\bar{J}(\mu, \nu_2)]. \quad (3.10)$$

The intuition of the construction is the following: At time $t = 0$, player 2, knowing Y_0 , can choose at random a decision $d \in \{1, 2\}$ such that the conditional law of Y_0 given that $d = 1$ is q_1 and the conditional law of Y_0 given $d = 2$ is q_2 . He will then play ν_1 if $d = 1$ and ν_2 when $d = 2$. More precisely, we set

$$\nu(\omega, u) = \sum_{\ell=1}^L \mathbb{1}_{Y_0=\ell} \left(\mathbb{1}_{u \in [0, \frac{\lambda(q_1)_\ell}{q_\ell}]} \nu_1(\omega, \frac{q_\ell}{\lambda(q_1)_\ell} u) + \mathbb{1}_{u \in (\frac{\lambda(q_1)_\ell}{q_\ell}, 1]} \nu_2(\omega, \frac{q_\ell u - \lambda(q_1)_\ell}{(1 - \lambda)(q_2)_\ell}) \right) \quad (3.11)$$

By the definition of μ the probability to choose ν_1 given that $Y_0 = \ell$ is $\frac{\lambda(q_1)_\ell}{q_\ell}$ whenever $q_\ell > 0$ and the probability to choose ν_2 is $\frac{(1-\lambda)(q_2)_\ell}{q_\ell}$. It follows that

$$\begin{aligned} \mathbb{E}_{p,q} [\bar{J}(\mu, \nu)] &= \mathbb{E}_{p,q} \left[\int_0^1 J(\mu, \nu(\cdot, u)) du \right] \\ &= \mathbb{E}_{p,q} \left[\sum_{\ell \in L} \mathbb{1}_{Y_0=\ell} \left(\int_0^{\frac{\lambda(q_1)_\ell}{q_\ell}} J(\mu, \nu_1(\cdot, \frac{q_\ell}{\lambda(q_1)_\ell} u)) \right. \right. \\ &\quad \left. \left. + \int_{\frac{\lambda(q_1)_\ell}{q_\ell}}^1 J(\mu, \nu_2(\cdot, \frac{q_\ell u - \lambda(q_1)_\ell}{(1 - \lambda)(q_2)_\ell})) du \right) \right] \\ &= \mathbb{E}_{p,q} \left[\sum_{\ell \in L} \mathbb{1}_{Y_0=\ell} \left(\frac{\lambda(q_1)_\ell}{q_\ell} \bar{J}(\mu, \nu_1) + \frac{(1 - \lambda)(q_2)_\ell}{q_\ell} \bar{J}(\mu, \nu_2) \right) \right] \\ &= \lambda \mathbb{E}_{p,q_1} [\bar{J}(\mu, \nu_1)] + (1 - \lambda) \mathbb{E}_{p,q_2} [\bar{J}(\mu, \nu_2)]. \end{aligned}$$

Maximizing (3.10) over $\mu \in \mathcal{T}_r^X$ yields then, using the ε optimality of ν_1 and ν_2 ,

$$V^+(p, q) \leq \lambda V^+(p, q_1) + (1 - \lambda) V^+(p, q_2) + \varepsilon$$

and the convexity in q follows since ε can be chosen arbitrarily small. \square

Next we define the concave conjugate in p of V^+ as

$$\forall x \in \mathbb{R}^K, q \in \Delta(L), V^{+,*}(x, q) := \inf_{p \in \Delta(K)} \{ \langle x, p \rangle - V^+(p, q) \}$$

and the convex conjugate in q of V^- as

$$\forall p \in \Delta(K), y \in \mathbb{R}^L, V_*^-(p, y) := \sup_{q \in \Delta(L)} \{ \langle q, y \rangle - V^-(p, q) \}.$$

Immediately from the previous lemma it follows that:

$$\begin{aligned} f^*(x, q) &\leq V^{+,*}(x, q) \leq h^*(x, q), \\ f_*(p, y) &\leq V_*^-(p, y) \leq h_*(p, y), \end{aligned} \quad (3.12)$$

where the functions h^*, f^* and h_*, f_* are defined as

$$\begin{aligned} h^*(x, q) &:= \inf_{p \in \Delta(K)} \{\langle x, p \rangle - h(p, q)\}, \quad f^*(x, q) := \inf_{p \in \Delta(K)} \{\langle x, p \rangle - f(p, q)\}, \\ h_*(p, y) &:= \sup_{q \in \Delta(L)} \{\langle q, y \rangle - h(p, q)\}, \quad f_*(p, y) := \sup_{q \in \Delta(L)} \{\langle q, y \rangle - f(p, q)\}. \end{aligned} \quad (3.13)$$

The next lemma provides an alternative formulation.

Lemma 3.6. *We have the following, alternative representations:*

$$\forall x \in \mathbb{R}^K, q \in \Delta(L), \quad V^{+,*}(x, q) = \sup_{\nu \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]) \quad (3.14)$$

$$\forall p \in \Delta(K), y \in \mathbb{R}^L, \quad V_*^-(p, y) = \inf_{\mu \in \mathcal{T}_r^X} \sup_{\nu \in \mathcal{T}^Y} \sup_{q \in \Delta(L)} (\langle y, q \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]). \quad (3.15)$$

Proof. Using remark 3.4, we have

$$V^{+,*}(x, q) = \inf_{p \in \Delta(K)} \sup_{\nu \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]).$$

Then, we will apply Fan's minmax theorem (see [13]) to deduce that:

$$\begin{aligned} V^{+,*}(x, q) &= \inf_{p \in \Delta(K)} \sup_{\nu \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]) \\ &= \sup_{\nu \in \mathcal{T}_r^Y} \inf_{p \in \Delta(K)} \inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]). \end{aligned}$$

In order to apply Fan's minmax theorem, we have to check that the function

$$(p, \nu) \in \Delta(K) \times \mathcal{T}_r^Y \rightarrow \inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)])$$

is concave-like with respect to $\nu \in \mathcal{T}_r^Y$ and affine (hence continuous) with respect to p in the compact convex set $\Delta(K)$.

To prove the concave-like property, given $\nu_1, \nu_2 \in \mathcal{T}_r^Y$, and $\lambda \in (0, 1)$, we define the mixed stopping time ν by

$$\nu(\omega, u) = \nu_1(\omega, \frac{u}{\lambda}) \mathbb{1}_{u \in [0, \lambda]} + \nu_2(\omega, \frac{u - \lambda}{1 - \lambda}) \mathbb{1}_{u \in [\lambda, 1]}.$$

A simple change of variables gives:

$$\begin{aligned} &\inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]) \\ &\geq \lambda \inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu_1)]) + (1 - \lambda) \inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu_2)]), \end{aligned}$$

which is exactly the concave-like property we need to apply Fan's theorem.

The second property follows from the relation:

$$\inf_{\mu \in \mathcal{T}^X} (\langle x, p \rangle - \mathbb{E}_{p,q}[\bar{J}(\mu, \nu)]) = \langle x, p \rangle - \sum_{k \in K} p^k \sup_{\mu \in \mathcal{T}^X} \mathbb{E}_{\delta_k, q}[\bar{J}(\mu, \nu)].$$

This last equation is proved in the same way as equation (3.9) in Lemma 3.5. \square

3.3 Dynamic Programming for $V^{+,*}$, V_*^-

We will now prove a dynamic programming inequality for $V^{+,*}$ and V_*^- .

Proposition 3.7. *For all $\varepsilon > 0$*

$$V^{+,*}(x, q) \geq \inf_{t \in [0, \varepsilon]} \left(e^{-rt} h^*(x_t, q_t) \mathbb{1}_{t < \varepsilon} + e^{-r\varepsilon} V^{+,*}(x_\varepsilon, q_\varepsilon) \mathbb{1}_{t = \varepsilon} \right), \quad (3.16)$$

where the dynamic (x_t, q_t) is given by

$$\forall t \geq 0, \quad x_t = x + \int_0^t (rI - R)x_s ds \quad \text{and} \quad q_t = q + \int_0^t {}^\top Q q_s ds.$$

Similarly, for all $\varepsilon' > 0$

$$V_*^-(x, q) \leq \sup_{t \in [0, \varepsilon']} \left(e^{-rt} f_*(p_t, y_t) \mathbb{1}_{s < \varepsilon'} + e^{-r\varepsilon'} V_*^-(p_{\varepsilon'}, y_{\varepsilon'}) \mathbb{1}_{t = \varepsilon'} \right), \quad (3.17)$$

with (p_t, y_t) given by

$$\forall t \geq 0, \quad p_t = p + \int_0^t {}^\top R p_s ds \quad \text{and} \quad y_t = x + \int_0^t (rI - Q)y_s ds.$$

Proof. In order to prove the dynamic programming inequalities, we need to recall the definition of the shift operator θ^X on Ω_X . For all $t \geq 0$, the map $\theta_t^X : \Omega_X \rightarrow \Omega_X$ is defined by

$$\forall s \geq 0, \quad \theta_t^X(\omega_X)(s) = \omega_X(s + t).$$

The shift operator θ^Y on Ω_Y is defined similarly.

We only give the proof of (3.16), the proof of (3.17) being similar. Given $\varepsilon > 0$, we consider the family $\mathcal{T}_{r,\varepsilon}^Y$ of mixed stopping times $\nu \in \mathcal{T}_r^Y$ such that there exists a mixed stopping time $\nu' \in \mathcal{T}_r^Y$ and $\nu(\omega, v) = \varepsilon + \nu'(\theta_\varepsilon^Y(\omega_Y), v)$.

As $\mathcal{T}_{r,\varepsilon}^Y \subset \mathcal{T}_r^Y$, we have

$$\begin{aligned} V^{+,*}(x, q) &\geq \sup_{\nu \in \mathcal{T}_{r,\varepsilon}^Y} \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \left(\langle x, p \rangle - \mathbb{E}_{p,q} \left[e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu < \varepsilon} \right. \right. \\ &\quad \left. \left. + e^{-r\varepsilon} \int_0^1 (e^{-r(\nu-\varepsilon)} f(X_\nu, Y_\nu) \mathbb{1}_{\nu < \mu} + e^{-r(\mu-\varepsilon)} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \nu}) dv \mathbb{1}_{\mu \geq \varepsilon} \right] \right). \end{aligned} \quad (3.18)$$

Let us fix $\nu \in \mathcal{T}_{r,\varepsilon}^Y$ (or equivalently $\nu' \in \mathcal{T}_r^Y$), $\mu \in \mathcal{T}^X$ and $p \in \Delta(K)$. By conditioning, we obtain:

$$\begin{aligned} &\mathbb{E}_{p,q} \left[e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} \int_0^1 (e^{-r(\nu-\varepsilon)} f(X_\nu, Y_\nu) \mathbb{1}_{\nu < \mu} + e^{-r(\mu-\varepsilon)} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \nu}) dv \mathbb{1}_{\mu \geq \varepsilon} \right] \\ &= \mathbb{E}_{p,q} \left[e^{-r\mu} \mathbb{E}_{p,q} [h(X_\mu, Y_\mu) | \mathcal{F}_\mu^X] \mathbb{1}_{\mu < \varepsilon} \right. \\ &\quad \left. + e^{-r\varepsilon} \mathbb{E}_{p,q} \left[\int_0^1 (e^{-r(\nu-\varepsilon)} f(X_\nu, Y_\nu) \mathbb{1}_{\nu < \mu} + e^{-r(\mu-\varepsilon)} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \nu}) dv | \mathcal{F}_\varepsilon^{X,Y} \right] \mathbb{1}_{\mu \geq \varepsilon} \right] \end{aligned}$$

Recall that the stopping time μ of the filtration \mathcal{F}^X can be identified with a stopping time defined on Ω_X . It is well-known that on the event $\mu \geq \varepsilon$, we have $\mu = \mu'(\omega_X, \theta_\varepsilon^X(\omega_X)) + \varepsilon$, where μ' is $\mathcal{F}_\varepsilon^X \otimes \mathcal{F}_\infty^X$ measurable and for all ω , $\mu'(\omega, \cdot)$ is an \mathcal{F}^X stopping time (see theorem 103 p. 151 in [10]). Then, using the Markov property, we deduce that

$$\begin{aligned} \mathbb{1}_{\mu \geq \varepsilon} \mathbb{E}_{p,q} \left[\int_0^1 \left(e^{-r(\nu-\varepsilon)} f(X_\nu, Y_\nu) \mathbb{1}_{\nu < \mu} + e^{-r(\mu-\varepsilon)} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \nu} \right) dv \middle| \mathcal{F}_\varepsilon^{X,Y} \right] \\ = \mathbb{1}_{\mu \geq \varepsilon} \mathbb{E}_{\delta_{X_\varepsilon}, \delta_{Y_\varepsilon}} [\bar{J}(\mu'(\omega, \cdot), \nu')]. \end{aligned}$$

On the other hand, since X and Y are independent we have

$$\mathbb{E}_{p,q}[\delta_{X_t, Y_t} | \mathcal{F}_t^X] = \delta_{X_t} \otimes q_t \in \Delta(K \times L). \quad (3.19)$$

Using the usual properties of the optional projection (see e.g. [10]) the previous equality implies $\mathbb{E}_{p,q}[h(X_\mu, Y_\mu) | \mathcal{F}_\mu^X] = h(\delta_{X_\mu}, q_\mu)$, and inequality (3.18) may be rewritten as

$$\begin{aligned} V^{+,*}(x, q) \geq \sup_{\nu' \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \\ \left(\langle x, p \rangle - \mathbb{E}_{p,q}[e^{-r\mu} h(\delta_{X_\mu}, q_\mu) \mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} \mathbb{E}_{X_\varepsilon, Y_\varepsilon}[\bar{J}(\mu'(\omega, \cdot), \nu')] \mathbb{1}_{\mu \geq \varepsilon}] \right). \end{aligned}$$

Defining

$$\forall (\nu', p, q) \in \mathcal{T}_r^Y \times \Delta(K) \times \Delta(L), \quad F(\nu', p, q) = \sup_{\hat{\mu} \in \mathcal{T}^X} \mathbb{E}_{p,q}[\bar{J}(\hat{\mu}, \nu')]$$

we have, using the same arguments as for (3.7) :

$$F(\nu', p, q) = \sum_{\ell \in L} \sum_{k \in K} q^\ell p^k F(\nu', \delta_k, \delta_\ell).$$

The previous inequality implies therefore

$$\begin{aligned} V^{+,*}(x, q) \geq \sup_{\nu' \in \mathcal{T}_r^Y} \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \\ \left(\langle x, p \rangle - \mathbb{E}_{p,q}[e^{-r\mu} h(\delta_{X_\mu}, q_\mu) \mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} F(\nu', \delta_{X_\varepsilon}, \delta_{Y_\varepsilon}) \mathbb{1}_{\mu \geq \varepsilon}] \right) \quad (3.20) \end{aligned}$$

and taking conditional expectation with respect to $\mathcal{F}_\varepsilon^X$, we obtain

$$\mathbb{E}_{p,q}[F(\nu', \delta_{X_\varepsilon}, \delta_{Y_\varepsilon}) \mathbb{1}_{\mu \geq \varepsilon} | \mathcal{F}_\varepsilon^X] = F(\nu', \delta_{X_\varepsilon}, q_\varepsilon) \mathbb{1}_{\mu \geq \varepsilon}.$$

Next, we apply the optional sampling theorem with the \mathcal{F}^X -stopping time $\mu \wedge \varepsilon$ and obtain

$$\begin{aligned} \langle x, p \rangle &= \mathbb{E}_{p,q}[\langle x, e^{-\mu \wedge \varepsilon} \delta_{X_{\mu \wedge \varepsilon}} \rangle \mathbb{1}_{\mu < \varepsilon} + \langle x, e^{-\varepsilon} \delta_{X_\varepsilon} \rangle \mathbb{1}_{\mu \geq \varepsilon}] \\ &= \mathbb{E}_{p,q}[\langle e^{-\mu \wedge \varepsilon} x, \delta_{X_{\mu \wedge \varepsilon}} \rangle \mathbb{1}_{\mu < \varepsilon} + \langle e^{-\varepsilon} x, \delta_{X_\varepsilon} \rangle \mathbb{1}_{\mu \geq \varepsilon}]. \end{aligned}$$

Substituting the last two equalities in the right-hand side of (3.20) yields

$$\begin{aligned} \langle x, p \rangle - \mathbb{E}_{p,q}[e^{-r\mu} h(\delta_{X_\mu}, q_\mu) \mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} F(\nu', \delta_{X_\varepsilon}, q_\varepsilon) \mathbb{1}_{\mu \geq \varepsilon}] \\ = \mathbb{E}_{p,q}[e^{-r\mu} (\langle e^{\mu(rI-R)} x, \delta_{X_\mu} \rangle - h(\delta_{X_\mu}, q_\mu)) \mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} (\langle e^{\varepsilon(rI-R)} x, \delta_{X_\varepsilon} \rangle - F(\nu', \delta_{X_\varepsilon}, q_\varepsilon)) \mathbb{1}_{\mu \geq \varepsilon}] \\ = \mathbb{E}_{p,q}[e^{-r\mu} (\langle x_\mu, \delta_{X_\mu} \rangle - h(\delta_{X_\mu}, q_\mu)) \mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon} (\langle x_\varepsilon, \delta_{X_\varepsilon} \rangle - F(\nu', \delta_{X_\varepsilon}, q_\varepsilon)) \mathbb{1}_{\mu \geq \varepsilon}]. \end{aligned}$$

Given any $\eta > 0$, let us choose ν' as an η -optimal minimizer in the problem $V^{+,*}(x_\varepsilon, q_\varepsilon)$ (note that these dynamics do not depend on p or μ), so that

$$\begin{aligned} (\langle x_\varepsilon, \delta_{X_\varepsilon} \rangle - F(\nu', \delta_{X_\varepsilon}, q_\varepsilon)) &\geq \inf_{p' \in \Delta(K)} (\langle x_\varepsilon, p' \rangle - F(\nu', p', q_\varepsilon)) \\ &= \inf_{p' \in \Delta(K)} \inf_{\hat{\mu} \in \mathcal{T}^X} (\langle x_\varepsilon, p' \rangle - \mathbb{E}_{p', q_\varepsilon}[\bar{J}(\hat{\mu}, \nu')]) \\ &\geq V^{+,*}(x_\varepsilon, q_\varepsilon) - \eta. \end{aligned}$$

Using the preceding results in (3.20), we deduce that for all $\eta > 0$

$$\begin{aligned} V^{+,*}(x, q) &\geq \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \mathbb{E}_{p, q}[e^{-r\mu}(\langle x_\mu, \delta_{X_\mu} \rangle - h(\delta_{X_\mu}, q_\mu))\mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon}(V^{+,*}(x_\varepsilon, q_\varepsilon) - \eta)\mathbb{1}_{\mu \geq \varepsilon}] \\ &\geq \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \mathbb{E}_{p, q}[e^{-r\mu}h^*(x_\mu, q_\mu)\mathbb{1}_{\mu < \varepsilon} + e^{-r\varepsilon}(V^{+,*}(x_\varepsilon, q_\varepsilon) - \eta)\mathbb{1}_{\mu \geq \varepsilon}]. \end{aligned}$$

Note that in the preceding expression, only μ is random, and thus we may replace the expectation with an integral with respect to the law of μ on $\overline{\mathbb{R}}_+$ denoted $P_{p, \mu}$ which yields

$$V^{+,*}(x, q) \geq \inf_{\mu \in \mathcal{T}^X} \inf_{p \in \Delta(K)} \int_{\overline{\mathbb{R}}_+} (e^{-rt}h^*(x_t, q_t))\mathbb{1}_{t < \varepsilon} + e^{-r\varepsilon}(V^{+,*}(x_\varepsilon, q_\varepsilon) - \eta)\mathbb{1}_{t \geq \varepsilon} dP_{p, \mu}(t).$$

Using the linearity of the integral and arguing as in Remark 3.4, the infimum over all admissible distributions for μ is equal to the infimum over Dirac masses (constant stopping times), and the conclusion follows by sending η to zero. \square

3.4 Subsolution property for V^+

We will prove the subsolution property for V^+ by establishing a super-solution property for $V^{+,*}$. The results rely on classical tools of convex analysis.

Notation 3.8. Let $C \subset \mathbb{R}^K$ and $D \subset \mathbb{R}^L$ denote two convex sets. For any $g : C \times D \rightarrow \mathbb{R}$ and $(x, y) \in C \times D$, we denote the sub-differential of g with respect to the first variable x by

$$\partial_1^- g(x, y) = \{x^* \in \mathbb{R}^K \mid \forall x' \in C, g(x, y) + \langle x^*, x' - x \rangle \leq g(x', y)\}.$$

The super-differential $\partial_1^+ g(x, y)$ is defined similarly.

We use the index ∂_2^- (resp. ∂_2^+), whenever we consider derivatives with respect to the second variable $y \in D$. We use ∂^+ , ∂^- for the full super- and sub-differential.

Proposition 3.9. For all $(x, q) \in \mathbb{R}^K \times \Delta(L)$, we have

$$\min\{(h^* - V^{+,*})(x, q); \vec{D}V^{+,*}(x, q; (rI - R)x, {}^\top Qq) - rV^{+,*}(x, q)\} \leq 0. \quad (3.21)$$

For all $(p, y) \in \Delta(K) \times \mathbb{R}^L$, we have

$$\max\{(f_* - V_*^-)(x, q); \vec{D}V_*^-(p, y; {}^\top Rp, (rI - Q)y) - rV_*^-(p, y)\} \geq 0. \quad (3.22)$$

Proof. We only show the first statement. Recall that by Proposition 3.7, for all $\varepsilon > 0$

$$V^{+,*}(x, q) \geq \inf_{t \in [0, \varepsilon]} (e^{-rt} h^*(x_t, q_t) \mathbb{1}_{t < \varepsilon} + e^{-r\varepsilon} V^{+,*}(x_\varepsilon, q_\varepsilon) \mathbb{1}_{t = \varepsilon}), \quad (3.23)$$

where the dynamic (x_t, q_t) is given by $x_t = x + \int_0^t (rI - R)x_s ds$ and $q_t = q + \int_0^t Qq_s ds$. We know that $h^*(x, q) - V^{+,*}(x, q) \geq 0$ by construction. In case $h^*(x, q) - V^{+,*}(x, q) > 0$, there exists by continuity an $\tilde{\varepsilon} > 0$ such that for all $0 < \varepsilon \leq \tilde{\varepsilon}$, choosing $t < \varepsilon$ would not be optimal in (3.23). Thus

$$V^{+,*}(x, q) \geq e^{-r\varepsilon} V^{+,*}(x_\varepsilon, q_\varepsilon). \quad (3.24)$$

We deduce that (3.21) holds since:

$$\vec{D}V^{+,*}(x, q; (rI - R)x, {}^\top Qq) - rV^{+,*}(x, q) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (e^{-r\varepsilon} V^{+,*}(x_\varepsilon, q_\varepsilon) - V^{+,*}(x, q)) \leq 0.$$

The last equality follows from the fact that $V^{+,*}$ is Lipschitz, implying that:

$$V^{+,*}(x_\varepsilon, q_\varepsilon) - V^{+,*}(x + \varepsilon(rI - R)x, q + \varepsilon {}^\top Qq) = o(\varepsilon).$$

□

Proposition 3.10. V^+ is a subsolution of (3.1). V^- is a supersolution of (3.2).

Proof. Again it is sufficient to prove only the subsolution property for V^+ , as the proof of the supersolution property for V^- is similar due to the symmetry of the problem. Since by construction $V^+(\bar{p}, \bar{q}) \leq f(\bar{p}, \bar{q})$ it remains to show that for $\bar{q} \in \Delta(L)$ and $\bar{p} \in \text{Ext}(V(., \bar{q}))$

$$V^+(\bar{p}, \bar{q}) > h(\bar{p}, \bar{q})$$

implies

$$rV^+(\bar{p}, \bar{q}) - \vec{D}_1 V^+(\bar{p}, \bar{q}; {}^\top R\bar{p}) - \vec{D}_2 V^+(\bar{p}, \bar{q}; {}^\top Q\bar{q}) \leq 0. \quad (3.25)$$

We first assume that \bar{p} is an exposed point of $V(., \bar{q})$ (see Definition A.3) and $V^+(\bar{p}, \bar{q}) > h(\bar{p}, \bar{q})$. We will reformulate (3.25) using the conjugate function $V^{+,*}$. Let us choose $\bar{x} \in \partial_1^+ V^+(\bar{p}, \bar{q})$ and $\bar{y} \in \partial_2^- V^+(\bar{p}, \bar{q})$ (see Lemma A.1), such that

$$\vec{D}_1 V^+(\bar{p}, \bar{q}; {}^\top R\bar{p}) = \langle \bar{x}, {}^\top R\bar{p} \rangle, \quad \vec{D}_2 V^+(\bar{p}, \bar{q}; {}^\top Q\bar{q}) = \langle \bar{y}, {}^\top Q\bar{q} \rangle.$$

By construction $V^+(\bar{p}, \bar{q}) = \langle \bar{x}, \bar{p} \rangle - V^{+,*}(\bar{x}, \bar{q})$ and (3.25) can be written as

$$\langle (rI - R)\bar{x}, \bar{p} \rangle - \langle {}^\top Q\bar{q}, \bar{y} \rangle - rV^{+,*}(\bar{x}, \bar{q}) \leq 0. \quad (3.26)$$

As \bar{p} is an exposed point, we know that there exists some $\hat{x} \in \partial_1^+ V^+(\bar{p}, \bar{q})$ such that in the expression

$$V^{+,*}(\hat{x}, \bar{q}) = \inf_{p \in \Delta(K)} \langle \hat{x}, p \rangle - V^+(p, \bar{q}),$$

the minimum is uniquely attained in \bar{p} . It follows that denoting $u := \hat{x} - \bar{x}$, for all $\varepsilon > 0$, the minimum in the expression

$$V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \inf_{p \in \Delta(K)} \langle \bar{x} + \varepsilon u, p \rangle - V^+(p, \bar{q}),$$

is uniquely attained in \bar{p} . Note that it may be that $\bar{x} = \hat{x}$ in which case $u = 0$. Fenchel's lemma implies that the function $V^{+,*}(\cdot, \bar{q})$ is differentiable at $\bar{x} + \varepsilon u$ with a gradient equal to \bar{p} and we have

$$V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \langle \bar{x} + \varepsilon u, \bar{p} \rangle - V^+(\bar{p}, \bar{q}).$$

Instead of proving directly (3.26), we will prove that for all $\varepsilon > 0$

$$\langle (rI - R)\bar{x} + \varepsilon u, \bar{p} \rangle - \langle {}^\top Q \bar{q}, \bar{y} \rangle - rV^{+,*}(\bar{x} + \varepsilon u, \bar{q}) \leq 0, \quad (3.27)$$

(3.26) follows then by sending $\varepsilon > 0$ to zero.

In order to apply Proposition 3.9, let us prove that $V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) < h^*(\bar{x} + \varepsilon u, \bar{q})$. By construction, we have $V^{+,*} \leq h^*$ since $V^+ \geq h$. Assume by contradiction that $V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = h^*(\bar{x} + \varepsilon u, \bar{q})$. Both functions being concave with respect to their first argument and since $D_1 V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \bar{p}$, we would have

$$D_1 h^*(\bar{x} + \varepsilon u, \bar{q}) = D_1 V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \bar{p},$$

and therefore

$$V^+(\bar{p}, \bar{q}) = \langle \bar{x} + \varepsilon u, \bar{p} \rangle - V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \langle \bar{x} + \varepsilon u, \bar{p} \rangle - h^*(\bar{x} + \varepsilon u, \bar{q}) = h(\bar{p}, \bar{q}),$$

which contradicts the assumption $V^+(\bar{p}, \bar{q}) - h(\bar{p}, \bar{q}) > 0$.

Note that $q \rightarrow V^+(\bar{p}, q)$ is convex on $\Delta(L)$ and that \bar{y} was chosen so that the directional derivative verifies (see Lemma A.1 for the first equality)

$$\vec{D}_2 V^+(\bar{p}, \bar{q}; {}^\top Q \bar{q}) = \max_{v \in \partial_2^- V^+(\bar{p}, \bar{q})} \langle v, {}^\top Q \bar{q} \rangle = \langle \bar{y}, {}^\top Q \bar{q} \rangle.$$

Since $V^{+,*}$ is a concave Lipschitz function on $\mathbb{R}^K \times \Delta(L)$, the envelope theorem (see Lemma A.2) implies

$$\partial^+ V^{+,*}(\bar{x} + \varepsilon u, \bar{q}) = \{\bar{p}\} \times (-\partial_2^- V^+(\bar{p}, \bar{q})). \quad (3.28)$$

Indeed, note that the right-hand side of (3.28) is the superdifferential of the concave function

$$(x, q) \in \mathbb{R}^K \times \Delta(L) \rightarrow \langle x, \bar{p} \rangle - V^+(\bar{p}, q),$$

at $(\bar{x} + \varepsilon u, \bar{q})$. We deduce that the directional derivatives of $V^{+,*}$ at $(\bar{x} + \varepsilon u, \bar{q})$ verify

$$\begin{aligned} \vec{D} V^{+,*}(\bar{x} + \varepsilon u, \bar{q}; (rI - R)\bar{x}, {}^\top Q \bar{q}) &= \min_{(w, v) \in \partial^+ V^{+,*}(\bar{x} + \varepsilon u, \bar{q})} \langle w, (rI - R)\bar{x} \rangle + \langle v, {}^\top Q \bar{q} \rangle \\ &= \langle \bar{p}, (rI - R)\bar{x} \rangle - \langle \bar{y}, {}^\top Q \bar{q} \rangle. \end{aligned}$$

(3.27) follows then from Proposition 3.9.

It remains to extend the result from exposed points to extreme points. We know that exposed points are a dense subset of extreme points (see Theorem 18.6 in [30]). We may therefore use an approximation argument by using Lemma A.4 applied to $p \rightarrow -V(\cdot, \bar{q})$ together with the fact that $p \rightarrow \vec{D}_2 V^+(p, \bar{q})$ is upper semi-continuous (use Lemma A.1 together with the fact that the correspondence of superdifferentials has closed graph). \square

3.5 Comparison principle

In the previous section we showed that $V^+(p, q)$ is a sub-solution to (3.1) while $V^-(p, q)$ verifies the super-solution property (3.2). Since $V^+(p, q) \geq V^-(p, q)$ by construction the following comparison principle will imply Theorem 3.3.

Let us recall the classical definition of an extreme point for a convex set.

Definition 3.11. *Let $C \subset \mathbb{R}^n$ a convex set. $x \in C$ is an extreme point of C if for any $x_1, x_2 \in C$ and $\lambda \in [0, 1]$:*

$$\lambda x_1 + (1 - \lambda)x_2 = x \Rightarrow x_1 = x_2 = x.$$

Theorem 3.12. *Let w_1, w_2 be two Lipschitz, concave-convex functions defined on $\Delta(K) \times \Delta(L)$ such that w_1 verifies the sub-solution property (3.1) and w_2 verifies the super-solution property (3.2). Then $w_1 \leq w_2$.*

Proof. We proceed by contradiction. Assume that $M := \max_{\Delta(K) \times \Delta(L)} w_1 - w_2 > 0$ and let C denote the compact set of $(p, q) \in \Delta(K) \times \Delta(L)$ where the maximum is reached. Let $(\bar{p}, \bar{q}) \in C$ denote an extreme point of the convex hull of C , which exists by Krein-Milman theorem and belongs to C by definition of the convex hull. It follows that \bar{p} is an extreme point of $w_1(\cdot, \bar{q})$ and that \bar{q} is an extreme point of $w_1(\bar{p}, \cdot)$.

Let us prove this property for \bar{p} (the case of \bar{q} being symmetric). Assume that there exists $p_1, p_2 \in \Delta(K)$ and $\lambda \in (0, 1)$ such that $\bar{p} = \lambda p_1 + (1 - \lambda)p_2$ and $\lambda w_1(p_1, \bar{q}) + (1 - \lambda)w_1(p_2, \bar{q}) = w_1(\bar{p}, \bar{q})$. Using that $w_2(\cdot, \bar{q})$ is concave, we would have

$$\lambda(w_1(p_1, \bar{q}) - w_2(p_1, \bar{q})) + (1 - \lambda)(w_1(p_2, \bar{q}) - w_2(p_2, \bar{q})) \geq w_1(\bar{p}, \bar{q}) - w_2(\bar{p}, \bar{q}) = M.$$

As $w_1 - w_2 \leq M$, we deduce that (p_1, \bar{q}) and (p_2, \bar{q}) belong to C and therefore that $p_1 = p_2$.

At point (\bar{p}, \bar{q}) , we have $f(\bar{p}, \bar{q}) \geq w_1(\bar{p}, \bar{q}) > w_2(\bar{p}, \bar{q}) \geq h(\bar{p}, \bar{q})$ so that

$$rw_1(\bar{p}, \bar{q}) - \vec{D}_1 w_1(\bar{p}, \bar{q}; {}^\top R \bar{p}) - \vec{D}_2 w_1(\bar{p}, \bar{q}; {}^\top Q \bar{q}) \leq 0,$$

$$rw_2(\bar{p}, \bar{q}) - \vec{D}_1 w_2(\bar{p}, \bar{q}; {}^\top R \bar{p}) - \vec{D}_2 w_2(\bar{p}, \bar{q}; {}^\top Q \bar{q}) \geq 0.$$

Note that ${}^\top R p$ (resp. ${}^\top Q q$) always belong to the tangent cone of $\Delta(K)$ at p (resp. of $\Delta(L)$ at q) so that directional derivatives are well-defined and real-valued. We deduce that

$$\begin{aligned} \vec{D}_1 w_1(\bar{p}, \bar{q}; {}^\top R \bar{p}) + \vec{D}_2 w_1(\bar{p}, \bar{q}; {}^\top Q \bar{q}) &\geq rw_1(\bar{p}, \bar{q}) \geq rw_2(\bar{p}, \bar{q}) + rM \\ &\geq \vec{D}_1 w_2(\bar{p}, \bar{q}; {}^\top R \bar{p}) + \vec{D}_2 w_2(\bar{p}, \bar{q}; {}^\top Q \bar{q}) + rM. \end{aligned}$$

It follows that one of the following inequalities holds true:

$$\vec{D}_1 w_1(\bar{p}, \bar{q}; {}^\top R \bar{p}) > \vec{D}_1 w_2(\bar{p}, \bar{q}; {}^\top R \bar{p})$$

$$\vec{D}_2 w_1(\bar{p}, \bar{q}; {}^\top Q \bar{q}) > \vec{D}_2 w_2(\bar{p}, \bar{q}; {}^\top Q \bar{q}).$$

In the first case, this would imply that for a sufficiently small ε ,

$$w_1(\bar{p} + \varepsilon {}^\top R \bar{p}, \bar{q}) - w_2(\bar{p} + \varepsilon {}^\top R \bar{p}, \bar{q}) > w_1(\bar{p}, \bar{q}) - w_2(\bar{p}, \bar{q}) = M,$$

and thus a contradiction. The second case is similar and this concludes the proof. \square

4 Optimal stopping times

Using the PDE characterization of the value function $V : \Delta(K) \times \Delta(L) \rightarrow \mathbb{R}$ by Theorem 3.3, it is possible to give a verification theorem for mixed stopping times for both players. Here we present the characterization of optimal stopping times μ for Player 1, i.e. the player observing X . By symmetry of the problem the result for Player 2 is given in a similar way.

In order to characterize optimal stopping times μ for Player 1, we introduce the belief process of the uninformed player over X . Let us fix a mixed stopping time $\mu \in \mathcal{T}_r^X$. Despite the fact that Player 2 has no information on X , he can compute for any $t \geq 0$ the conditional distribution π_t (defined below) of X_t given the event that player 1 did not stop before time t . To that end we consider the product probability space

$$(\Omega', \mathcal{F}', \mathbb{P}'_{p,q}) := (\Omega_X \times \Omega_Y \times [0, 1], \mathcal{F}_\infty^X \otimes \mathcal{F}_\infty^Y \otimes \mathcal{B}([0, 1]), \mathbb{P}_{p,q} \otimes \text{Leb}),$$

where Leb stands for the Lebesgue measure. The stopping time $\mu(\omega_X, u)$ is thus seen as a random variable defined on Ω' .

We define the belief process π taking values in $\Delta(K)$ as a càdlàg version of:

$$(\mathbb{P}'_{p,q}[X_t = k | \mathcal{H}_t^\mu])_{k \in K, t \geq 0}, \quad (4.1)$$

where \mathcal{H}^μ is the usual right-continuous augmentation of $\sigma(\mathbb{1}_{\mu \leq s}, 0 \leq s \leq t)$. By construction, the process π has the following property:

$$\forall 0 \leq s \leq t, \mathbb{E}_{\mathbb{P}'_{p,q}}[\pi_t | \mathcal{F}_s^\pi] = e^{(t-s)^\top R} \pi_s. \quad (4.2)$$

We will now focus on a specific type of stopping times μ , those which induce a belief process π which can be extended on the time interval $[0, \mu]$ to a piecewise deterministic Markov process (PDMP) $Z = (\pi, \xi)$ having specific structural properties, where ξ is an auxiliary dual variable playing the role of a subgradient in $\partial_2^- V_*$.

Let us now describe precisely the type of PDMP we will consider (see Davis [9] for more details). Let E denote a closed subset of $\Delta(K) \times \mathbb{R}^L$ and let (α, λ, ϕ) be given with:

- $\alpha : E \rightarrow \mathbb{R}^K \times \mathbb{R}^L$: locally bounded measurable vector field such that for all initial point $z \in E$, there exists a unique global solution of $w'_z(t) = \alpha(w_z(t))$ with initial condition $w_z(0) = z$ which stays in E .
- $\lambda : E \rightarrow \mathbb{R}_+$: locally bounded measurable intensity function.
- $\phi : E \rightarrow E$: locally bounded measurable jump function such that for all $z \in E$, $\phi(z) \neq z$
- We assume in addition that for all $z \in E$, the maps $t \rightarrow \alpha(w_z(t)), \lambda(w_z(t)), \phi(w_z(t))$ are càdlàg.

The construction of the PDMP Z with characteristics (α, λ, ϕ) and initial position $Z_0 = z$ is as follows: Let T_1 denote a non-negative random variable such that $\mathbb{P}(T_1 > t) = \exp(-\int_0^t \lambda(w_z(s)) ds)$. Then, define $Z_t = w_z(t)$ for $t \in [0, T_1)$ and $Z_{T_1} = \phi(w_z(T_1))$. For $k \geq 2$, construct by induction the variables $T_k - T_{k-1}$, Z_{T_k} , and the process Z on $[T_{k-1}, T_k]$ using the same method where z is replaced by $Z_{T_{k-1}}$.

The process Z has locally Lipschitz trajectories on the intervals $[T_{k-1}, T_k)$ and jumps at times T_k . We also assume that for all $z \in E$ and all $t \geq 0$, $\mathbb{E}[\sum_{s \leq t} |Z_s - Z_{s-}|] < \infty$. This condition ensures that the sequence T_k goes to $+\infty$ so that the process Z is well-defined for all times.

We now consider the following two closed subsets of $\Delta(K) \times \mathbb{R}^L$

$$\mathcal{H} := \{(p, y) \in \Delta(K) \times \mathbb{R}^L \mid rV_*(p, y) - \vec{D}V_*(p, y; {}^\top R p, (rI - Q)y) \geq 0\},$$

$$\mathcal{S} := \{(p, y) \in \Delta(K) \times \mathbb{R}^L \mid V_*(p, y) = h_*(p, y)\}.$$

We now introduce a set of assumptions, called structure conditions on E and (α, λ, ϕ) that will be used in our verification theorem below.

Definition 4.1. *We say that E and (α, λ, ϕ) fulfill the structure conditions (SC) if*

(SC1) $E = E_{\mathcal{H}} \cup \mathcal{S}$ with $E_{\mathcal{H}}$ a nonempty closed subset of \mathcal{H} , and $E_{\mathcal{H}} \cap \mathcal{S} = \emptyset$.

(SC2) For all $z \in \mathcal{S}$, $\lambda(z) = 0$ and $\alpha(z) = 0$. For all $z \in E_{\mathcal{H}}$, for all $t \geq 0$, $w_z(t) \in E_{\mathcal{H}}$.

(SC3) $\phi : E_{\mathcal{H}} \rightarrow \mathcal{S}$ and for all $z \in E_{\mathcal{H}}$, if we denote $\phi(z) = (\phi_p(z), \phi_y(z))$, then

$$\{k \in K \mid p_k = 0\} \subset \{k \in K \mid \phi_p(z)_k = 0\}.$$

(SC4) For all $z \in E_{\mathcal{H}}$:

$$\lambda(z) \left(V_*(\phi(z)) - V_*(z) - \vec{D}V_*(z; \phi(z) - z) \right) = 0. \quad (4.3)$$

$$\vec{D}V_*(z; \alpha(z)) + \vec{D}V_*(z; \lambda(z)(\phi(z) - z)) = \vec{D}V_*(z; \alpha(z) + \lambda(z)(\phi(z) - z)). \quad (4.4)$$

$$\alpha(z) + \lambda(z)(\phi(z) - z) = Az \text{ with } A = \begin{pmatrix} {}^\top R & 0 \\ 0 & rI - Q \end{pmatrix} \quad (4.5)$$

(SC5) For all $z \in (\Delta(K) \times \mathbb{R}^L) \setminus E$, there exists $z' \in E_{\mathcal{H}}$, $z'' \in \mathcal{S}$ and $m \in [0, 1]$ such that

$$z = (1 - m)z' + mz'', \quad V_*(z) = (1 - m)V_*(z') + mV_*(z'').$$

Comments:

- Condition (SC2) means that all points in \mathcal{S} are absorbing and that $E_{\mathcal{H}}$ is invariant by the flow associated to the ODE $w' = \alpha(w)$.
- Condition (SC3) means that the direction of jump is deterministic and that any jump should start in $E_{\mathcal{H}}$ and end in \mathcal{S} . (SC2) and (SC3) together ensure that the PDMP Z with characteristics (α, λ, ϕ) is well-defined and integrable for any starting point in E .
- Condition (4.3) states that jumps occur only over the flat parts of the graph of V_* .
- Since V_* is convex, condition (4.4) means that for all z , V_* is differentiable at z in the direction of the cone generated by $\alpha(z)$ and the vector $\lambda(z)(\phi(z) - z)$. Note that this is automatically true when $\lambda(z) = 0$ as V_* admits directional derivatives.

- Condition (4.5) was designed in such a way that if Z is a PDMP with characteristics (α, λ, ϕ) with $Z_0 = z \in \mathcal{H}$, then

$$\forall t \geq 0, \mathbb{E}[Z_{t \wedge T_1}] = \mathbb{E}\left[\int_0^{t \wedge T_1} AZ_r dr\right]. \quad (4.6)$$

Indeed, according to Theorem 5.5 in Davis [9], the identity map belongs to the extended generator of the process Z so that Dynkin's formula applies, implying that:

$$\begin{aligned} \mathbb{E}[Z_{t \wedge T_1}] &= \mathbb{E}\left[\int_0^{t \wedge T_1} (\alpha(Z_r) + \lambda(Z_r)(\phi(Z_r) - Z_r)) dr\right] \\ &= \mathbb{E}\left[\int_0^{t \wedge T_1} AZ_r dr\right]. \end{aligned}$$

- Condition (SC5) means that for any point z outside of E , there exists a random initial condition Z_0 for the process, such that Z_0 takes only two values, one in \mathcal{S} and one in $E_{\mathcal{H}}$, such that $\mathbb{E}[Z_0] = z$ and $\mathbb{E}[V_*(Z_0)] = V_*(z)$.

We are now ready to state our verification Theorem.

Theorem 4.2. *Let $(p, q) \in \Delta(K) \times \Delta(L)$ and $y \in \partial_2^- V(p, q)$. If there exists characteristics (α, λ, ϕ) fulfilling conditions (SC), then the stopping time $\mu \in \mathcal{T}_r^X$ defined below is optimal.*

Case 1: *If $z = (p, y) \in E_{\mathcal{H}}$, let us denote $z_t = (p_t, y_t) = w_z(t)$.*

Recall that (ω, u) denotes the canonical element in Ω' and that u is independent of ω and uniformly distributed over $[0, 1]$. Let us consider a sequence of independent variables $(U_n(u))_{n \geq 0}$ uniformly distributed over $[0, 1]$. Given $t \geq 0$ and $k \in K$, define as a measurable function of $\tilde{u} \in [0, 1]$ the unique non-decreasing map $M(t, k)(\tilde{u})$ with values in $[0, +\infty]$ such that:

$$\int_0^1 \mathbb{1}_{\{M(t, k)(\tilde{u}) \geq x\}} d\tilde{u} = \exp\left(-\int_t^{t+x} \frac{\phi_p(z_s)_k}{(p_s)_k} \lambda(z_s) ds\right),$$

with the convention $\frac{0}{0} = 0$. Let $(S_n)_{n \geq 1}$ denote the sequence of jumps of the Markov chain X and let $S_0 = 0$. Define, with the convention $\inf \emptyset = +\infty$:

$$\mu_n(\omega, u) = \inf\{t \in [S_n, S_{n+1}) \mid t - S_n \geq M(S_n, X_{S_n})(U_n(u))\}.$$

Then $\mu_z(\omega, u) = \inf_{n \geq 0} \mu_n(\omega, u)$ is optimal.

Case 2: *If $z \notin E$, let $z' = (p', y')$, $z'' = (p'', y'')$ and $m \in (0, 1)$ be given by condition (SC5). Then an optimal stopping time is given by:*

$$\mu(\omega, u) = 0 \cdot \mathbb{1}_{u \leq m, X_0 \in \text{supp}(p')} + \mu_{z'}(\omega, \frac{u - m}{1 - m}) \mathbb{1}_{u > m},$$

where $\mu_{z'}$ is the stopping time constructed in case 1 and $\text{supp}(p')$ denotes the support of p' .

Case 3: *If $z \in \mathcal{S}$, $\mu = 0$ is optimal.*

Comments:

- Note that we do not impose any extra regularity assumption on the function V_* . We actually conjecture that this verification theorem can always be applied for the model we consider, but the proof of such a conjecture (i.e. of the existence of (α, λ, ϕ) having the required properties) would rely on complex methods of convex analysis and geometry and is beyond the scope of this paper.
- The construction of the characteristics (α, λ, ϕ) is illustrated through two examples in section 5 where a precise description of the optimal stopping time μ is given.

Proof. Part 1: We construct a process Z which is an \mathcal{H}^μ -PDMP with state space $E \subset \Delta(K) \times \mathbb{R}^L$ and characteristics (α, λ, ϕ) . Some details are omitted as this first part rely on classical probabilistic arguments. Throughout the proof, \mathbb{P} stands for $\mathbb{P}'_{p,q}$, and \mathbb{E} for the associated expectation operator.

- We consider at first the case $z \in E_{\mathcal{H}}$.

Define the belief process π by (4.1). Let us denote $z_t = (p_t, y_t) = w_z(t)$. Define then the process ξ by

$$\forall t \geq 0, \xi_t = y_t \mathbb{1}_{t < \mu} + \phi_y(z_\mu) \mathbb{1}_{t \geq \mu}.$$

Note that ξ is an \mathcal{H}^μ -adapted c'adl'ag random process taking values in \mathbb{R}^L .

We will prove that the process Z with $Z_t := (\pi_{t \wedge \mu}, \xi_t)$ is an \mathcal{H}^μ -PDMP with state space $E \subset \Delta(K) \times \mathbb{R}^L$, characteristics (α, λ, ϕ) , having a unique jump at time μ .

Let $T := \sup\{t \mid \mathbb{P}(\mu > t) > 0\}$. By definition of μ , we have $T > 0$ and for all $k \in K$ and all $0 \leq t < T$ such that $(p_t)_k > 0$:

$$\frac{1}{h} \left(\mathbb{P}(\mu \in (t, t+h) \mid X_t = k, \mu > t) - \int_t^{t+h} \frac{\phi_p(z_s)_k}{(p_s)_k} \lambda(z_s) ds \right) \xrightarrow{h \rightarrow 0+} 0, \quad (4.7)$$

and the convergence is locally uniform in t . The proof is omitted as it follows from standard properties of jump processes.

We claim that:

$$\forall t \geq 0, \pi_{t \wedge \mu} = p_t \mathbb{1}_{t < \mu} + \pi_\mu \mathbb{1}_{t \geq \mu}. \quad (4.8)$$

Indeed, for all $t \in [0, T)$, on $\{t < \mu\}$, we have $\pi_t(k) = \mathbb{P}(X_t = k \mid \mu > t)$ so that we only have to check that

$$\mathbb{P}(X_t = k \mid \mu > t) = p_t(k).$$

Let $c(t)_k := \mathbb{P}(X_t = k \mid \mu > t)$, $b(t) := \mathbb{P}(\mu > t)$ for $t \in [0, T)$. By conditioning on the events $\{X_t = k, \mu > t\}$ for all $k \in K$, we have for all t in $[0, T)$:

$$\frac{1}{h} (b(t+h) - b(t)) \xrightarrow{h \rightarrow 0+} b'(t) = -b(t) \sum_k c(t)_k \frac{\phi_p(z_t)_k}{(p_t)_k} \lambda(z_t). \quad (4.9)$$

On the other hand, X is Markov with respect to the filtration $t \rightarrow \mathcal{F}_t^X \vee \sigma(u)$ and $\{\mu > t\}$ belongs to the completion of $\mathcal{F}_t^X \vee \sigma(u)$ (see e.g. Proposition 1 in [31]), so that the conditional law of $(X_s)_{s \geq t}$ given $\{\mu > t\}$ is $\mathbb{P}_{c(t)}$. We deduce that

$$\frac{1}{h} (\mathbb{P}(X_{t+h} = k \mid \mu > t) - \mathbb{P}(X_t = k \mid \mu > t)) \xrightarrow{h \rightarrow 0+} (\mathbb{R}c(t))_k.$$

We have:

$$\begin{aligned}
\frac{1}{h}(c(t+h)_k - c(t)_k) &= \frac{1}{h}[\mathbb{P}(X_{t+h} = k | \mu > t+h) - \mathbb{P}(X_{t+h} = k | \mu > t) \\
&\quad + \mathbb{P}(X_{t+h} = k | \mu > t) - \mathbb{P}(X_t = k | \mu > t)] \\
&= \frac{1}{h} \frac{1}{b(t)} (\mathbb{P}(X_{t+h} = k, \mu > t+h) - \mathbb{P}(X_{t+h} = k, \mu > t)) \\
&\quad + \frac{1}{h} \mathbb{P}(X_{t+h} = k, \mu > t+h) \left(\frac{1}{b(t+h)} - \frac{1}{b(t)} \right) \\
&\quad + \frac{1}{h} (\mathbb{P}(X_{t+h} = k | \mu > t) - \mathbb{P}(X_t = k | \mu > t))
\end{aligned}$$

Using that

$$\begin{aligned}
\mathbb{P}(X_{t+h} = k, \mu > t+h) - \mathbb{P}(X_{t+h} = k, \mu > t) &= -\mathbb{P}(X_{t+h} = k, \mu \in (t, t+h]) \\
&= -\mathbb{P}(X_t = k, \mu \in (t, t+h]) + o(h),
\end{aligned}$$

we deduce that for all $t \in [0, T)$:

$$\frac{1}{h} \frac{1}{b(t)} (\mathbb{P}(X_{t+h} = k, \mu > t+h) - \mathbb{P}(X_{t+h} = k, \mu > t)) \xrightarrow{h \rightarrow 0+} -c(t)_k \frac{\phi_p(z_t)_k}{(p_t)_k} \lambda(z_t).$$

We conclude that for all $t \in [0, T)$:

$$\frac{1}{h}(c(t+h)_k - c(t)_k) \xrightarrow{h \rightarrow 0+} c'(t)_k = (\mathbb{T}Rc(t))_k - \lambda(z_t) \left[\phi_p(z_t)_k \frac{c(t)_k}{(p_t)_k} - c(t)_k \sum_{k'} \frac{c(t)_{k'}}{(p_t)_{k'}} \phi_p(z_t)_{k'} \right].$$

Using condition (4.5), we can check that $t \rightarrow p_t$ is the unique solution of the above differential equation, so that $c(t) = p(t)$ for all $t \in [0, T)$ since $c(0) = p(0)$. This concludes the proof of (4.8). We deduce also that for all $t \geq 0$, we have:

$$b(t) = \mathbb{P}(\mu > t) = \exp\left(-\int_0^t \lambda(z_s) ds\right).$$

Indeed, using that $p_t = c(t)$, the differential equation in the right-hand side of (4.9) admits the above expression as unique solution.

We now prove that $\pi_\mu = \phi_p(z_\mu)$ on $\{\mu < \infty\}$. We start with the fact that

$$\mathbb{P}(X_t = k | \mu \in (t, t+h]) = -b(t)(p_t)_k \frac{\mathbb{P}(\mu \in (t, t+h] | X_t = k, \mu > t)}{b(t+h) - b(t)}.$$

For all $t \in [0, T)$, we have therefore:

$$\mathbb{P}(X_t = k | \mu \in (t, t+h]) \xrightarrow{h \rightarrow 0+} \phi_p(z_t)_k, \tag{4.10}$$

and the convergence is locally uniform in t .

For all $n \geq 1$ and all $i \in \mathbb{N}$, let $t_i^n = \frac{i}{2^n}$. Let \mathcal{H}^n denote the σ -field generated by the events $(\{\mu \in (t_i^n, t_{i+1}^n]\})_{i \geq 0}$. Then $\sigma(\mu) = \bigvee_{n \geq 0} \mathcal{H}^n$, and (up to null sets) $\mathcal{H}_\mu^\mu = \sigma(\mu)$. For all $k \in K$, we have by the martingale convergence theorem:

$$(\pi_\mu)_k \mathbb{1}_{\mu < \infty} = \mathbb{P}(X_\mu = k | \mathcal{H}_\mu^\mu) \mathbb{1}_{\mu < \infty} = \lim_{n \rightarrow \infty} \mathbb{P}(X_\mu = k | \mathcal{H}^n) \mathbb{1}_{\mu < \infty} a.s.$$

On the other hand, for all $n \geq 0$, we have:

$$\mathbb{P}(X_\mu = k | \mathcal{H}^n) \mathbb{1}_{\mu < \infty} = \sum_{i \geq 0: t_i^n < T} \mathbb{1}_{\mu \in (t_i^n, t_{i+1}^n]} \mathbb{P}(X_\mu = k | \mu \in (t_i^n, t_{i+1}^n]).$$

The following convergence holds locally uniformly in t for $t \in [0, T)$:

$$\left(\mathbb{P}(X_\mu = k | \mu \in (t, t + \frac{1}{2^n}]) - \mathbb{P}(X_t = k | \mu \in (t, t + \frac{1}{2^n}]) \right) \xrightarrow{n \rightarrow \infty} 0.$$

We deduce easily that the following convergence holds almost surely:

$$\mathbb{P}(X_\mu = k | \mathcal{H}^n) \mathbb{1}_{\mu < \infty} \rightarrow \phi_p(z_\mu)_k \mathbb{1}_{\mu < \infty},$$

which concludes the proof that $\pi_\mu = \phi_p(z_\mu)$ on $\{\mu < \infty\}$.

To summarize, the process Z is such that for all $t \geq 0$

$$Z_t = z_t \mathbb{1}_{t < \mu} + \phi(z_\mu) \mathbb{1}_{t \geq \mu},$$

and μ is such that $\mathbb{P}(\mu > t) = \exp(-\int_0^t \lambda(z_s) ds)$. As, up to null sets, \mathcal{H}^μ is the natural filtration of Z , we deduce that Z is an \mathcal{H}^μ -PDMP with characteristics (α, λ, ϕ) , having a unique jump at time μ .

• If $z \notin E$, define the belief process π by (4.1). Then $\pi_0 = z' \mathbb{1}_{\mu > 0} + z'' \mathbb{1}_{\mu = 0}$ and $\mathbb{P}(\mu > 0) = 1 - m$. Let us denote $z_t = (p_t, y_t) = w_{z'}(t)$. Define then the process ξ by

$$\forall t \geq 0, \xi_t = y_t \mathbb{1}_{t < \mu} + \phi_y(z_\mu) \mathbb{1}_{t \geq \mu}.$$

As for the previous case, the process Z with $Z_t := (\pi_{t \wedge \mu}, \xi_t)$ is an \mathcal{H}^μ -PDMP with state space $E \subset \Delta(K) \times \mathbb{R}^L$, characteristics (α, λ, ϕ) , having a unique jump at time μ . Indeed, reasoning conditionally on the event $\{\mu > 0\}$, the same analysis as above shows that for all $t \geq 0$

$$Z_t \mathbb{1}_{\mu > 0} = z_t \mathbb{1}_{0 < t < \mu} + \phi(z_\mu) \mathbb{1}_{t \geq \mu > 0},$$

and that μ is such that $\mathbb{P}(\mu > t | \mu > 0) = \exp(-\int_0^t \lambda(z_s) ds)$ for all $t \geq 0$. On the other hand, on the event $\{\mu = 0\}$, we have $Z_t = Z_0 = z'' \in \mathcal{S}$ for all $t \geq 0$, which concludes the proof.

• Case $z \in \mathcal{S}$. The process defined by $Z_t = z$ for all $t \geq 0$, is an \mathcal{H}^μ -PDMP with state space $E \subset \Delta(K) \times \mathbb{R}^L$, characteristics (α, λ, ϕ) .

Part 2: We prove the optimality of μ . Thanks to the results of part 1, we have constructed an \mathcal{H}^μ -PDMP $Z_t = (\pi_t, \xi_t)$ with state space $E \subset \Delta(K) \times \mathbb{R}^L$ and characteristics (α, λ, ϕ) such that

(i) Z_0 takes finitely many values and:

$$\mathbb{E}[Z_0] = (p, y), \quad \mathbb{E}[V_*(Z_0)] = V_*(p, y).$$

(ii) $Z_t \in \mathcal{H}$ on $[0, \mu)$, Z may jump only at time μ and $Z_\mu \in \mathcal{S}$.

As V_* is convex and Lipschitz, and the vector field α is locally bounded, the map $t \rightarrow V_*(w_{\tilde{z}}(t))$ is locally Lipschitz for all $\tilde{z} \in E$, and therefore absolutely continuous with right-derivative $\vec{D}V_*(w_{\tilde{z}}(t); \alpha(w_{\tilde{z}}(t)))$. It follows that V_* belongs to the extended generator of the PDMP Z (see Theorem 5.5 in Davis [9]) and Dynkin's formula gives for any $t \in \mathbb{R}_+$

$$\begin{aligned} V_*(p, y) &= \mathbb{E}[V_*(Z_0)] \\ &= \mathbb{E}[e^{-r(\mu \wedge t)} V_*(Z_{\mu \wedge t}) + \int_0^{\mu \wedge t} r e^{-rs} V_*(\pi_s, \xi_s) ds \\ &\quad - \int_0^{\mu \wedge t} e^{-rs} (\vec{D}V_*(Z_s; \alpha(Z_s)) + \lambda(Z_s)(V_*(\phi(z)) - V_*(Z_s))) ds] \end{aligned} \quad (4.11)$$

Using properties (SC4), we deduce that:

$$\begin{aligned} V_*(p, y) &= \mathbb{E}[e^{-r(\mu \wedge t)} V_*(Z_{\mu \wedge t}) + \int_0^{\mu \wedge t} e^{-rs} (r V_*(\pi_s, \xi_s) - \vec{D}V_*(Z_s; \alpha(Z_s)) \\ &\quad + \vec{D}V_*(Z_s; \lambda(Z_s)(\phi(Z_s) - Z_s))) ds] \\ &= \mathbb{E}[e^{-r(\mu \wedge t)} V_*(Z_{\mu \wedge t}) \\ &\quad + \int_0^{\mu \wedge t} e^{-rs} (r V_*(Z_s) - \vec{D}V_*(Z_s; \alpha(Z_s) + \lambda(Z_s)(\phi(Z_s) - Z_s))) ds] \\ &= \mathbb{E}[e^{-r(\mu \wedge t)} V_*(Z_{\mu \wedge t}) + \int_0^{\mu \wedge t} e^{-rs} (r V_*(Z_s) - \vec{D}V_*(Z_s; AZ_s)) ds]. \end{aligned}$$

By construction, we have that $r V_*(Z_s) - \vec{D}V_*(Z_s; AZ_s) \geq 0$ on $\{s < \mu\}$. This implies that for all t :

$$\begin{aligned} V_*(p, y) &\geq \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r(\mu \wedge t)} V_*(Z_{\mu \wedge t})] = \mathbb{E}_{\mathbb{P}'_{p,q}} [\mathbb{1}_{t < \mu} e^{-rt} V_*(Z_t) + \mathbb{1}_{\mu \leq t} e^{-r\mu} V_*(Z_\mu)] \\ &\geq \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-rt} f_*(Z_t) \mathbb{1}_{t < \mu} + e^{-r\mu} h_*(Z_\mu) \mathbb{1}_{\mu \leq t}] \\ &= \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-rt} f_*(X_t, \xi_t) \mathbb{1}_{t < \mu} + e^{-r\mu} h_*(X_\mu, \xi_\mu) \mathbb{1}_{\mu \leq t}], \end{aligned}$$

where the last equality follows by conditioning with respect to \mathcal{H}_t^μ and \mathcal{H}_μ^μ . Finally, we obtain

$$\begin{aligned} V_*(p, y) &\geq \sup_{t \in \mathbb{R}_+} \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-rt} f_*(X_t, \xi_t) \mathbb{1}_{t < \mu} + e^{-r\mu} h_*(X_\mu, \xi_\mu) \mathbb{1}_{\mu \leq t}] \\ &= \sup_{\tau \in \mathcal{T}^Y} \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} f_*(X_\tau, \xi_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h_*(X_\mu, \xi_\mu) \mathbb{1}_{\mu \leq \tau}], \end{aligned} \quad (4.12)$$

where the last equality follows from the fact that the expression inside the expectation is independent of \mathcal{F}_∞^Y (and using the same method as for Remark 3.4).

To conclude the proof, we use duality exactly as in Proposition 3.10.

Recall that since $y \in \partial_2^- V(p, q)$, we have

$$V(p, q) = \langle q, y \rangle - V_*(p, y).$$

It follows that

$$V(p, q) \leq \inf_{\tau \in \mathcal{T}^Y} \langle q, y \rangle - \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} f_*(X_\tau, \xi_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h_*(X_\mu, \xi_\mu) \mathbb{1}_{\mu \leq \tau}].$$

For any $\tau \in \mathcal{T}^Y$, we have, using that Y and ξ are independent

$$\langle q, y \rangle = \mathbb{E}_{p,q} [e^{-r\tau} \langle \delta_{Y_\tau}, \xi_\tau \rangle \mathbb{1}_{\tau < \mu} + e^{-r\mu} \langle \delta_{Y_\mu}, \xi_\mu \rangle \mathbb{1}_{\mu \leq \tau}].$$

We deduce that

$$\begin{aligned} \langle q, y \rangle - \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} f_*(X_\tau, \xi_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h_*(X_\mu, \xi_\mu) \mathbb{1}_{\mu \leq \tau}] \\ = \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} (\langle \delta_{Y_\tau}, \xi_\tau \rangle - f_*(X_\tau, \xi_\tau)) \mathbb{1}_{\tau < \mu} + e^{-r\mu} (\langle \delta_{Y_\mu}, \xi_\mu \rangle - h_*(X_\mu, \xi_\mu)) \mathbb{1}_{\mu \leq \tau}]. \end{aligned} \quad (4.13)$$

It follows from the definitions of f_*, h_* that:

$$\begin{aligned} \langle \delta_{Y_\tau}, \xi_\tau \rangle - f_*(X_\tau, \xi_\tau) &\leq f(X_\tau, Y_\tau) \\ \langle \delta_{Y_\mu}, \xi_\mu \rangle - h_*(X_\mu, \xi_\mu) &\leq h(X_\mu, Y_\mu), \end{aligned}$$

from which we conclude that

$$\begin{aligned} \langle q, y \rangle - \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} f_*(X_\tau, \xi_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h_*(X_\mu, \xi_\mu) \mathbb{1}_{\mu \leq \tau}] \\ \leq \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} f(X_\tau, Y_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \tau}]. \end{aligned} \quad (4.14)$$

Substituting in the previous inequality, we have

$$V(p, q) \leq \inf_{\tau \in \mathcal{T}^Y} \mathbb{E}_{\mathbb{P}'_{p,q}} [e^{-r\tau} f(X_\tau, Y_\tau) \mathbb{1}_{\tau < \mu} + e^{-r\mu} h(X_\mu, Y_\mu) \mathbb{1}_{\mu \leq \tau}] = \inf_{\tau \in \mathcal{T}^Y} \mathbb{E}_{\mathbb{P}_{p,q}} [\bar{J}(\mu, \tau)],$$

which proves the optimality of μ . \square

5 Two examples

In this section, we provide explicit solutions for two examples in order to illustrate how to apply our verification result.

5.1 Asymmetric Information with constant states

We assume that $K = L = \{0, 1\}$ and that $R = Q = 0$, which means that $X_t = X_0$ and $Y_t = Y_0$ almost surely for all $t \geq 0$.

With an abuse of notation, we write $V(p, q)$ for $V((p, 1-p), (q, 1-q))$ for $(p, q) \in [0, 1]^2$ (and similarly for f, h). V as well as f and h are thus seen as functions defined on $[0, 1]^2$.

Let us consider the particular case

$$h(p, q) = 3p + 2q - 4, \quad f(p, q) = 2p + 3q - 1.$$

More general cases can be solved with similar arguments.

5.1.1 The value function

By Theorem 3.3 we have that the value function V is the unique Lipschitz saddle function such that $f \geq V \geq h$ and

$$\forall q \in [0, 1], \forall p \in \text{Ext}(V(\cdot, q)), \quad V(p, q) > h(p, q) \Rightarrow rV(p, q) \leq 0, \quad (5.1)$$

$$\forall p \in [0, 1], \forall q \in \text{Ext}(V(p, \cdot)), \quad V(p, q) < f(p, q) \Rightarrow rV(p, q) \geq 0. \quad (5.2)$$

The explicit expression for V is given in the next proposition.

Proposition 5.1.

$$V(p, q) = \begin{cases} 0 & \text{if } (p, q) \in [\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\ \frac{2q-1}{q}(p+q-1) & \text{if } p \geq 1-q, \text{ and } q \geq \frac{1}{2} \\ \frac{1-2p}{1-p}(p+q-1) & \text{if } q \leq 1-p, \text{ and } p \leq \frac{1}{2}. \end{cases}$$

One may prove the above proposition by verification, by checking that V fulfills all the conditions given in Theorem 3.3. Nevertheless, let us explain briefly how to obtain directly this explicit expression. (5.1) implies that for any given $q \in [0, 1]$, the map $p \rightarrow V(p, q)$ is affine on any interval on which it is non-negative, and (5.2) implies a dual property for $q \rightarrow V(p, q)$. Using these properties, it is easy to check that

$$V(p, 0) = \begin{cases} f(p, 0) & \text{if } p \leq \frac{1}{2} \\ 0 & \text{if } p \geq \frac{1}{2} \end{cases}, \quad V(p, 1) = ph(1, 1),$$

$$V(1, q) = \begin{cases} 0 & \text{if } q \leq \frac{1}{2} \\ h(1, q) & \text{if } q \geq \frac{1}{2} \end{cases}, \quad V(0, q) = (1-q)f(0, 0).$$

Using that V is a saddle function, we deduce also that

$$\forall (p, q) \in \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \quad V(p, q) = 0.$$

Using continuity and concavity with respect to the first variable, there exists necessarily a one-to-one map $p \in [0, 1/2] \rightarrow m(p)$ such that $V(p, m(p)) = 0$ (it may be any closed set using only continuity of V , but concavity implies that it is indeed a curve). Using the above-mentioned properties, the following functions are affine

$$\forall q \in [1/2, 1], p \in [m^{-1}(q), 1] \rightarrow V(p, q),$$

$$\forall p \in [0, 1/2], q \in [0, m(p)] \rightarrow V(p, q).$$

Writing the condition that V is a saddle function at points $(p, m(p))$, it leads to $m(p) = 1 - p$.

Remark 5.2. *If one slightly perturbs the coefficients of the functions f, h , the above method still applies. However, the expression of the map m will be more complex.*

5.1.2 Non-existence of the value with non-randomized stopping times

Let us define \hat{V}^+, \hat{V}^- as the lower and upper value of the same game where both players are restricted to pure (classical) stopping times, i.e. for all $(p, q) \in [0, 1]^2$ (using the same convention as above)

$$\hat{V}^+(p, q) := \inf_{\nu \in \mathcal{T}^Y} \sup_{\mu \in \mathcal{T}^X} \mathbb{E}_{p, q} [\bar{J}(\mu, \nu)], \quad (5.3)$$

$$\hat{V}^-(p, q) := \sup_{\mu \in \mathcal{T}^X} \inf_{\nu \in \mathcal{T}^Y} \mathbb{E}_{p, q} [\bar{J}(\mu, \nu)]. \quad (5.4)$$

Then a direct computation shows that:

$$\begin{aligned}\hat{V}^-(p, q) &= \sup_{t_0, t_1 \in \overline{\mathbb{R}}_+} \inf_{s_0, s_1 \in \overline{\mathbb{R}}_+} \sum_{i, j=0,1} p_i q_j e^{-r(t_i \wedge s_j)} (h(i, j) \mathbb{1}_{t_i < s_j} + f(i, j) \mathbb{1}_{s_j \leq t_j}) \\ &= \begin{cases} pq - (1 - q) & \text{if } pq > (1 - q) \\ 0 & \text{if } p \geq 1/2 \text{ and } pq \leq 1 - q \\ (1 - 2p)(pq - (1 - q)) & \text{if } p < 1/2 \text{ and } pq \leq 1 - q \end{cases}\end{aligned}$$

where $p_0 = p$ and $p_1 = 1 - p$, and similarly for q_0, q_1 . On the other hand, by symmetry of the problem, one shows that:

$$\hat{V}^+(p, q) = -\hat{V}^-(1 - q, 1 - p).$$

The reader may check that $p \rightarrow \hat{V}^-(p, 2/3)$ is not concave, that $q \rightarrow \hat{V}^+(1/3, q)$ is not convex and that

$$\forall p \in (0, 1/2), \hat{V}^+(p, 1 - p) > V(p, 1 - p) > \hat{V}^-(p, 1 - p)$$

which proves that the value does not exist in general when players are restricted to pure stopping times.

5.1.3 Optimal strategies

We will show how to construct for our example an optimal stopping time $\mu \in \mathcal{T}_r^X$ for Player 1. Optimal stopping times for Player 2 can be determined in a similar way.

For $(p, y) \in [0, 1] \times \mathbb{R}$, we consider the (restricted) convex conjugate

$$V_*(p, y) := V_*((p, 1 - p), (y, 0)) = \max_{q \in [0, 1]} qy - V(p, q).$$

As $V_*((p, 1 - p), (y_1, y_2)) = y_2 + V_*((p, 1 - p), (y_1 - y_2, 0))$, Theorem 4.2 can be applied without any modification to the functions V and V_* as redefined here. Note that the set \mathcal{H} is given by

$$\mathcal{H} := \{(p, y) \in [0, 1] \times \mathbb{R} \mid rV_*(p, y) - \bar{D}_2 V_*(p, y; ry) \geq 0\}.$$

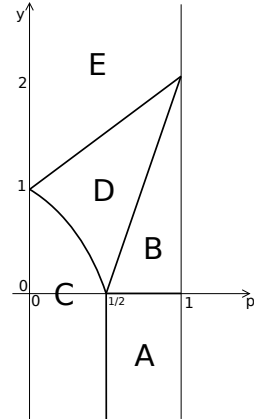
In order to calculate V_* we distinguish different zones of $[0, 1] \times \mathbb{R}$.

We set

$$\begin{aligned}A &= \{(p, y) : p \geq 1/2, y \leq 0\} \\ B &= \{(p, y) : p \geq 1/2, y \in [0, 4p - 2]\} \\ C &= \{(p, y) : p \leq 1/2, y \leq \frac{1-2p}{1-p}\} \\ D &= \{(p, y) : p \geq 1/2, y \in [4p - 2, 1 + p]\} \\ &\quad \cup \{(p, y) : p \leq 1/2, y \in [\frac{1-2p}{1-p}, 1 + p]\} \\ E &= \{(p, y) : y \geq 1 + p\}.\end{aligned}$$

A direct computation shows that

$$V_*(p, y) = \begin{cases} 0 & \text{for } (p, y) \in A \\ \frac{1}{2}y & \text{for } (p, y) \in B \\ 1 - 2p & \text{for } (p, y) \in C \\ -2\sqrt{2 - y}\sqrt{1 - p} + 3 - 2p & \text{for } (p, y) \in D \\ y - p & \text{for } (p, y) \in E, \end{cases}$$



and that

$$h_*(p, y) = \begin{cases} 4 - 3p & \text{if } y \leq 2 \\ y + 2 - 3p & \text{if } y \geq 2. \end{cases}$$

It follows that $\mathcal{S} = \{(p, y) \mid V * (p, y) = h_*(p, y)\} = \{(1, y) \mid y \geq 2\}$. One can easily check that

$$\mathcal{H} = A \cup B \cup C \cup \{p = 0\}.$$

Furthermore we note that V_* is affine on each zone A, B, C, E .

On D , V_* is not an affine function, however it is easily seen that for each $p \in [0, 1/2]$, V_* is affine on the segment joining the point $(p, \frac{1-2p}{1-p})$ and the point $(1, 2)$. In order to construct an optimal stopping time we shall first construct some characteristics (α, λ, ϕ) satisfying the conditions (SC) of Definition 4.1.

The actual state space of our PDMP Z will be $E = E_{\mathcal{H}} \cup \mathcal{S}$ with $E_{\mathcal{H}} = A \cup C \cup \{p = 0\}$.

Roughly speaking, when starting in $E_{\mathcal{H}}$, this process has to stay in $E_{\mathcal{H}}$ and may jump one time over a flat part of the graph of V_* from $E_{\mathcal{H}}$ to $E_{\mathcal{S}} = \mathcal{S}$. The condition (4.5) implies that for all $z = (p, y) \in E_{\mathcal{H}}$:

$$\lambda(z)(\phi(z) - z) + \alpha(z) = (0, ry). \quad (5.5)$$

As $E_{\mathcal{H}}$ has to be invariant for the flow associated to the vector field α , if for some point z , the solution of the differential equation associated to the vector field $(0, ry)$ starting at z does not exit $E_{\mathcal{H}}$ immediately, we simply choose $\alpha(z) = (0, ry)$ and $\lambda(z) = 0$.

We are left to consider the points $z = (p, \frac{1-2p}{1-p})$ for $p \in (0, 1/2)$. For these points, the process has to jump with positive intensity, otherwise it will exit from $E_{\mathcal{H}}$. In order for the jump to be on a flat part of the graph of V_* , the only possible ending point in \mathcal{S} is $(1, 2)$. We define therefore $\phi(z) = (1, 2)$. The condition (4.4) requires V_* to be differentiable at z in the direction of the cone generated by $\{\alpha(z), \lambda(z)((1, 2) - z)\}$. Together with the fact that the solution of $w' = \alpha(w)$ starting at z has to stay in $E_{\mathcal{H}}$, this implies that $\alpha(z)$ is tangent to the boundary of $E_{\mathcal{H}}$, i.e. $\alpha(z) = \kappa(z)(1, \frac{-1}{(1-p)^2})$ for some constant $\kappa(z)$.

Equation 5.5 becomes:

$$\lambda(z)((1, 2) - z) + \kappa(z)(1, \frac{-1}{(1-p)^2}) = (0, ry),$$

which leads to the unique solution

$$\lambda(z) = \frac{r(1-2p)}{2}, \quad \alpha(z) = \frac{r(1-2p)}{2} \left(-(1-p), \frac{1}{1-p} \right) = \left(-\frac{r}{2}(1-2p)(1-p), \frac{r}{2}y \right).$$

Below, we describe the optimal stopping time μ for all the possible pairs (p, y) given by Theorem 4.2.

Let $(p, y) \in \Delta(K) \times \mathbb{R}$.

- If $(p, y) \in \{y \leq 0\} \cup \{p = 0\}$, then $\mu = +\infty$ is optimal.
- If $(p, y) \in \mathcal{S}$, then $\mu = 0$ is optimal.
- If $(p, y) \in E \cap \{0 < p < 1\}$, then an optimal μ is given by:

$$\mu = 0.\mathbb{1}_{X_0=0} + (+\infty).\mathbb{1}_{X_0=1}.$$

The induced belief process is $\pi_t = 1.\mathbb{1}_{X_0=0} + 0.\mathbb{1}_{X_0=1}$ for $t \geq 0$, and ξ is defined by:

$$\forall t \geq 0, \xi_t = e^{rt}\xi_0, \text{ with } \xi_0 = 2.\mathbb{1}_{X_0=0} + \frac{y-2p}{1-p}\mathbb{1}_{X_0=1}.$$

• If $p \in (0, \frac{1}{2})$ and $y \in (0, \frac{1-2p}{1-p}]$, let $z_t = (p_t, y_t)$ denote the unique solution of $\frac{dz_t}{dt} = a(z_t)$ with initial condition $z_0 = (p, y)$. Then the optimal μ is given by

$$\mu(\omega, u) = (+\infty)\mathbb{1}_{X_0=1} + S(u)\mathbb{1}_{X_0=0},$$

where S is the unique non-decreasing function from $(0, 1)$ to \mathbb{R} such that

$$\forall t \geq 0, \int_0^1 \mathbb{1}_{S(u) > t} du = \exp\left(-\int_0^t \rho(z_s) ds\right),$$

with $\rho(z_s) = \frac{1}{p_s}\lambda(p_s, y_s)$. The induced belief process is given by:

$$\forall t \geq 0, \pi_t = p_t\mathbb{1}_{t < \mu} + 1.\mathbb{1}_{t \geq \mu},$$

and ξ is defined by:

$$\forall t \geq 0, \xi_t = y_t\mathbb{1}_{t < \mu} + 2e^{r(t-\mu)}\mathbb{1}_{t \geq \mu}.$$

Player 1 simply stops with some intensity $\rho(z_t) = \rho(Z_t)$ conditionally on the fact that $X_0 = 0$. The induced intensity of jump of Z is therefore equal to $\pi_t\rho(Z_t) = \lambda(Z_t)$ as required.

• If $(p, y) \in B$, then an optimal μ is defined by

$$\mu(\omega, u) = 0.\mathbb{1}_{\{X_0=0, u \leq \frac{y}{2p}\}} + (+\infty).\mathbb{1}_{\{X_0=1\} \cup \{X_0=0, u > \frac{y}{2p}\}}.$$

It means that Player 1 stops with probability $\frac{y}{2p}$ at time 0 conditionally on $X_0 = 0$, and never stops otherwise. The induced belief process is $\pi_t = \frac{2p-y}{2-y}\mathbb{1}_{\mu \leq t} + 1.\mathbb{1}_{\mu > t}$ for $t \geq 0$ and ξ is defined by $\xi_t = 0.\mathbb{1}_{\mu \leq t} + 2e^{rt}\mathbb{1}_{\mu > t}$ for $t \geq 0$.

• If $(p, y) \in D$, let $p' \in [0, 1/2]$ be such that (p, y) belongs to the segment joining $(p', \frac{1-2p'}{1-p'})$ and $(1, 2)$. An easy computation shows that

$$p' = 1 - \sqrt{\frac{1-p}{2-y}}.$$

Define $z_t = (p_t, y_t)$ as the unique solution of $\frac{dz_t}{dt} = \alpha(z_t)$ with initial condition $z_0 = (p', \frac{1-2p'}{1-p'})$. Then an optimal μ is defined by

$$\mu(\omega, u) = 0.\mathbb{1}_{\{X_0=0, u \leq x\}} + S\left(\frac{u-x}{1-x}\right)\mathbb{1}_{\{X_0=0, u > x\}} + (+\infty).\mathbb{1}_{\{X_0=1\}}.$$

where $x = \frac{p-p'}{p(1-p')}$ and S is the unique non-decreasing function from $(0, 1)$ to \mathbb{R} such that

$$\forall t \geq 0, \int_0^1 \mathbb{1}_{S(u) > t} du = \exp\left(-\int_0^t \rho(z_s) ds\right).$$

with $\rho_s = \frac{1}{p_s} \lambda(p_s, y_s)$. The induced belief process is given by:

$$\forall t \geq 0, \pi_t = p_t \mathbb{1}_{t < \mu} + 1 \cdot \mathbb{1}_{t \geq \mu},$$

and ξ is defined by:

$$\forall t \geq 0, \xi_t = y_t \mathbb{1}_{t < \mu} + 2e^{r(t-\mu)} \mathbb{1}_{t \geq \mu}.$$

Player 1 stops at time 0 conditionally on $X_0 = 0$ with probability x , and x was constructed so that $\pi_0 = p' \mathbb{1}_{\mu > 0} + 1 \cdot \mathbb{1}_{\mu = 0}$. If he did not stop at time 0, Player 1 stops with intensity $\rho(z_t) = \rho(Z_t)$ conditionally on $X_0 = 0$. As above, the induced intensity of jump for Z_t is $\pi_t \rho(Z_t) = \lambda(Z_t)$.

5.2 Case of incomplete information on one side

Next, we consider the particular case where the set L is reduced to a single point, implying that player 2 has no private information. In this context, admissible stopping times of player 2 are simply random times (see Definition 2.1) and the value function V depends only on $p \in \Delta(K)$.

Furthermore we assume that there are only two states for the Markov chain X which is observed by player 1, i.e. $K := \{0, 1\}$. We choose X to be an ergodic chain with generator $R := \begin{pmatrix} -a & a \\ b & -b \end{pmatrix}$, where $a, b > 0$ and the unique invariant probability is $(\frac{b}{a+b}, \frac{a}{a+b})$.

With a slight abuse of notation, we write $V(p)$ for $V(p, 1-p)$ $p \in [0, 1]$ (and similarly for f, h). V as well as f and h are thus seen as functions defined on $[0, 1]$. Rather than providing a complete study of this one-dimensional case, we work under the following assumptions:

(H1) For all $p \in [0, 1]$, $0 < h(p) < f(p)$.

(H2) f and h are increasing on $[0, 1]$, i.e. $h(0) < h(1)$ and $f(0) < f(1)$.

We impose (H1) and (H2) merely to simplify the presentation. Other cases can be studied with similar arguments.

5.2.1 Value function

By Theorem 3.3, V is the unique concave Lipschitz function on $[0, 1]$ such that $f \geq V \geq h$ and

$$\begin{aligned} \forall p \in [0, p^*], V(p) < f(p) &\Rightarrow rV(p) + ((a+b)p - b)V'_r(p) \geq 0 \\ \forall p \in (p^*, 1], V(p) < f(p) &\Rightarrow rV(p) - ((a+b)p - b)V'_l(p) \geq 0, \end{aligned}$$

and for any extreme point $p \in \text{Ext}(V)$:

$$\begin{aligned} p \in [0, p^*] \text{ and } V(p) > h(p) &\Rightarrow rV(p) + ((a+b)p - b)V'_r(p) \leq 0 \\ p \in (p^*, 1] \text{ and } V(p) > h(p) &\Rightarrow rV(p) - ((a+b)p - b)V'_l(p) \leq 0, \end{aligned}$$

where V'_l, V'_r denote the left and right derivative of V and $p^* = \frac{b}{a+b}$.

The following result can be shown by explicitly solving (5.6).

Proposition 5.3. *Under assumptions (H1) and (H2) we distinguish three cases:*

(i) if $\frac{b}{b+r}h(1) > f(0)$ the value function is given by

$$V(p) = \begin{cases} f(p) & \text{on } [0, p_0] \\ \frac{p-p_0}{1-p_0}h(1) + \frac{1-p}{1-p_0}f(p_0) & \text{on } (p_0, 1], \end{cases} \quad (5.7)$$

where p_0 is the unique solution in $(0, p^*)$ of the quadratic equation:

$$\frac{h(1) - f(p_0)}{1 - p_0} = \frac{-rf(p_0)}{(a+b)p_0 - b}. \quad (5.8)$$

(ii) if $h(0) < \frac{b}{b+r}h(1) \leq f(0)$, we have

$$V(p) = \frac{b}{b+r}h(1)(1-p) + ph(1).$$

(iii) if $\frac{b}{b+r}h(1) \leq h(0)$ we have $V = h$ on $[0, 1]$.

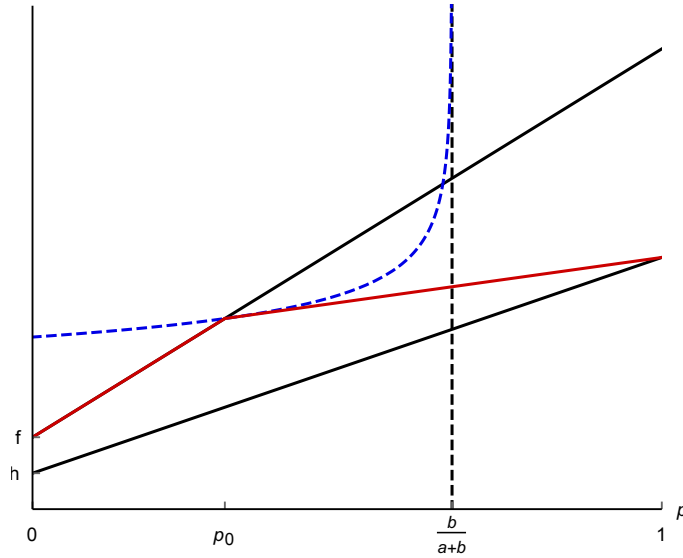


Figure 5.1: Value function in case (i)

The proof follows of Proposition 5.3 follows by verification by checking that (5.6) holds. One may also easily obtain the solution by a direct approach. Indeed, note that all positive solutions to

$$rw(p) + (b - (a+b)p)w'(p) = 0 \quad (5.9)$$

are strictly convex, so that using (H1) and (5.6), we may prove that $V(p)$ necessarily is a concave and piecewise affine function, having no extreme point in $\{h < V < f\} \cap (0, 1)$. Using then (H2) and (5.6) applied in $p = 1$ (which is always an extreme point), we deduce that $V(1) = h(1)$, and (5.6) applied in $p = 0$ allows to compute $V(0)$. The problem is solved in cases (ii) and (iii). For case (i), we have $V(0) = f(0)$, and (5.6) gives a condition for the left or right derivative of V in the only interior extreme point p_0 , which is (5.8). Figure 5.1

illustrates the shape of the function V (in red) in case (i). p_0 is the unique point in $(0, p^*)$ such that the solution of (5.9) passing through $(p_0, f(p_0))$ (the blue dotted line) is tangent to the line joining $(p_0, f(p_0))$ and $(1, h(1))$.

Remark 5.4. In case (i), the solution $S(p)$ of the game where none of the players observe the Markov chain X is easily obtained by solving the obstacle problem (3.4). Precisely, we obtain that $S = f$ on $[0, p_1]$, $S = h$ on $[p_2, 1]$ and that S solves the ODE

$$rS(p) + ((a+b)p - b)S'(p) = 0$$

on the interval $[p_1, p_2]$ where $p_1 < p_0 < p_2 < p^*$. The points p_1, p_2 are obtained using that S has to be differentiable at p_2 (the “smooth fit” condition). Note that despite the characterizations for S and V rely on the same variational inequalities, the convexity constraints for V and the fact that the variational inequalities apply only at extreme points imply that the shape of V significantly differs from that of S .

The solution in case (ii) implies that it is optimal for player 1 to wait for the more favorable state 0 before he stops, while the solution in case (iii) implies that it is optimal for player 1 to stop immediately. The optimal strategy for the more interesting case (i) is detailed in the next section.

5.2.2 Optimal Strategy for the informed player

We construct an optimal strategy for the player 1 in case that (H1) and (H2) and the condition (i) of Proposition 5.3 are valid. To construct optimal stopping times we will use Theorem 4.2. As the parameter $q \in \Delta(L)$ plays no role here (L is a singleton), we let the reader verify that Theorem 4.2 applies directly to the function $-V$ in place of V^* since formally $\Delta(L) = \{1\}$ and for $y \in \mathbb{R}$

$$V^*(p, y) = y - V(p).$$

As a result, the component ξ of the PDMP Z plays absolutely no role, and can be replaced by 0. In the following, we suppress the component ξ and we will construct a PDMP π having the required properties.

The reader can easily verify that the set \mathcal{H} is given by $\mathcal{H} = [0, p_0]$, where p_0 is given by (5.8) and that $\mathcal{S} = \{1\}$.

Let us construct the characteristics (α, λ, ϕ) of the PDMP π meeting the structure conditions (SC) of Definition 4.1. The state space of π will be $E = E_{\mathcal{H}} \cup \mathcal{S}$ with $E_{\mathcal{H}} = \mathcal{H}$. At first the structure condition (4.5) becomes

$$\forall p \in [0, p_0], \lambda(p)(\phi(p) - p) + \alpha(p) = -ap + b(1 - p).$$

We define $\alpha(p) = -ap + b(1 - p)$ and $\lambda(p) = 0$ for $p \in [0, p_0]$. For the point p_0 , we need to have a positive intensity of jump since the solution of $w'(t) = -aw(t) + b(1 - w(t))$ starting at p_0 exits \mathcal{H} immediately. Thanks to (SC3), the jump has to end in \mathcal{S} so that we define $\phi(p_0) = 1$. The above condition becomes

$$\alpha(p_0) = -ap_0 + b(1 - p_0) - \lambda(p_0)(1 - p_0).$$

The condition (4.4) requires V to be differentiable in the cone generated by $\alpha(p_0)$ and $\lambda(p_0)(1-p_0) > 0$. As V is not differentiable in p_0 , this implies $\alpha(p_0) \geq 0$ so that the condition becomes equivalent to the existence of a right-derivative. On the other hand, the fact that \mathcal{H} has to be invariant by the flow of the ODE $w' = \alpha(w)$ implies $\alpha(p_0) \leq 0$. The only possible choice is thus $\alpha(p_0) = 0$ and $\lambda(p_0) = \frac{b-(a+b)p_0}{1-p_0}$.

The next proposition describes optimal stopping times for all initial $p \in [0, 1]$ obtained by applying Theorem 4.2.

Proposition 5.5. *Assume that (H1) and (H2) and the condition (i) of Proposition 5.3 are valid. An optimal strategy $\mu : \Omega \times [0, 1] \rightarrow [0, +\infty)$ for Player 1 is given by the following expressions.*

(i) For $p = p_0$:

$$\mu(\omega, u) := \inf \left\{ t \geq 0 \mid \int_0^t \mathbb{1}_{X_s=0} ds \geq \frac{-\ln(1-u)}{\lambda_1} \right\}, \text{ with } \lambda_1 := \frac{b-(a+b)p_0}{p_0(1-p_0)} > 0.$$

That means Player 1 stops with intensity λ_1 conditionally on $X_t = 0$.

(ii) For $p > p_0$:

$$\mu(\omega, u) := \begin{cases} 0 & \text{if } X_0 = 0 \text{ and } u \in [0, c(p)) \\ \inf \left\{ t \geq 0 \mid \int_0^t \mathbb{1}_{X_s=0} ds \geq \frac{-\ln(1-\tilde{u})}{\lambda_1} \right\} & \text{otherwise,} \end{cases}$$

where $c(p) := \frac{p-p_0}{p(1-p_0)}$ and $\tilde{u} := \frac{u-c(p)}{1-c(p)} \mathbb{1}_{X_0=0} + u \mathbb{1}_{X_0=1}$.

That means that at time $t = 0$, Player 1 stops with probability $\frac{p-p_0}{p(1-p_0)}$ if $X_0 = 0$, and if he did not stop at $t = 0$, he stops with intensity λ_1 conditionally on the fact that $X_t = 0$.

(iii) For $p < p_0$:

$$\mu(\omega, u) := \inf \left\{ t \geq t(p) \mid \int_{t(p)}^t \mathbb{1}_{X_s=0} ds \geq \frac{-\ln(1-u)}{\lambda_1} \right\},$$

where $t(p) := \inf \{ t \geq 0 \mid \mathbb{P}_p(X_t = 0) = p_0 \}$.

That means that Player 1 waits until the probability that Player 2 assigns to the event $X_t = 0$ reaches p_0 and then stops with intensity λ_1 conditionally on the fact that $X_t = 0$.

Note that the same method can be applied to compute explicitly optimal strategies for Player 2 in this example. These solutions being quite similar to the previously studied cases, they are not presented here.

6 Open questions

Several open problems arise. A natural problem is to investigate the case where the Markov process X, Y have infinite state space, e.g. diffusion processes. The main difficulty is that our approach leads formally to study partial differential equations in infinite dimensional spaces of probabilities. The only results in this direction consider differential games where the information parameters of the game do not evolve over time (see [7]).

Very interesting question is whether our methods can be adapted to consider Markov chains (X, Y) which are correlated. Indeed, in our proof of the Dynamic Programming principle the independence plays an important role, since it allows to detach the two dynamics in (3.19). The case of correlated information in the information asymmetry is static has been studied in [27]. Its generalization to an evolving setting is far from being obvious and an interesting subject for further studies.

In view of possible applications to stopping games arising e.g. in financial mathematics, one may also consider models with publicly observed diffusive dynamics. The particular case where the information parameters were not evolving was considered in [19]. A generalization of these results is an interesting point for further research.

A Technical proofs and auxiliary tools

A.1 Auxiliary results of convex analysis

We prove here some elementary results in convex analysis that we are using in the proof of Proposition 3.9. The following lemmas are easy adaptations of classical and well-known results to Lipschitz convex functions with polyhedral domains. As references covering exactly what we need were difficult to find, we decided to add this appendix for the convenience of the reader. Note that we do not try to provide the most general version of these lemmas.

Lemma A.1. *Let f be a Lipschitz convex function from C to \mathbb{R} . Assume that C is a polyhedron, then if $T_C(x)$ denotes the tangent cone of C at x :*

$$\forall x \in C, \forall v \in T_C(x), \vec{D}f(x; v) = \max_{u \in \partial^- f(x)} \langle u, v \rangle.$$

Proof. This relation holds for any point x in the relative interior of C (see Theorem 23.4 in [30]). For x in the relative boundary of C , the left hand-side of the above equality is always greater than the right-hand side using the definition of subgradients.

For $m \in \mathbb{N}^*$ with $m \geq M$, where M is the Lipschitz constant of f , define the Moreau-Yosida regularization

$$\forall x \in \mathbb{R}^n, f_m(x) := \inf_{y \in C} f(y) + m|y - x|.$$

The function f_m is convex, m -Lipschitz and coincides with f on C . As $f_m \leq f$, we have for all $x \in C$, $\partial^- f_m(x) \subset \partial^- f(x)$. For any $v \in T_C(x)$, we have that $x + tv \in C$ for all sufficiently small $t > 0$ (here we use the polyhedron assumption) and therefore:

$$\vec{D}f(x; v) = \vec{D}f_m(x; v) = \sup_{u \in \partial^- f_m(x)} \langle u, v \rangle \leq \sup_{u \in \partial^- f(x)} \langle u, v \rangle.$$

□

Lemma A.2 (Danskin). *Let P be a non-empty compact subset of \mathbb{R}^m and C a non-empty polyhedron in \mathbb{R}^n . Let f be a real-valued Lipschitz function defined on $P \times C$. Assume that for all $p \in P$, the function f_p defined on \mathbb{R}^n by $f_p(x) = f(p, x)$ is a convex function. Define*

$g(x) = \sup_{p \in P} f(p, x)$. Then, if for $\bar{x} \in C$, the maximum $\max_{p \in P} f(p, \bar{x})$ is uniquely attained in \bar{p} , we have

$$\partial^- g(\bar{x}) = \partial^- f_{\bar{p}}(\bar{x}).$$

Proof. Note that both sets are non-empty due to the Lipschitz assumption, and that the inclusion $\partial^- f_{\bar{p}}(\bar{x}) \subset \partial^- g(\bar{x})$ is a direct consequence of the definitions. Let us prove the reverse inclusion. Assume by contradiction that there exists $v \in \partial^- g(\bar{x}) \setminus \partial^- f_{\bar{p}}(\bar{x})$. Then, using a separation argument, there exists $z \in \mathbb{R}^n$ and $\varepsilon > 0$ such that:

$$\forall u \in \partial^- f_{\bar{p}}(\bar{x}), \langle z, v \rangle \geq \varepsilon + \langle z, u \rangle. \quad (\text{A.1})$$

For all $u \in \partial^- f_{\bar{p}}(\bar{x})$, w in the normal cone of C at \bar{x} and $j \in \mathbb{N}^*$, we have $u + jw \in \partial^- f_{\bar{p}}(\bar{x})$. Replacing u by $u + jw$ in (A.1), and taking the limit as $j \rightarrow \infty$, we deduce that $\langle z, w \rangle \leq 0$, implying that z belongs to the tangent cone of C at \bar{x} . Since C is a polyhedron, $\bar{x} + tz \in C$ for all $t \in (0, \alpha)$ for some $\alpha > 0$. Up to replace z by αz , we may assume that $\bar{x} + z \in C$. Let p_n be a sequence maximizing $f(p, \bar{x} + \frac{z}{n})$. The sequence p_n converges to \bar{p} using the continuity of f and g . For $s \in (0, 1)$ and n such that $\frac{1}{n} \leq s$ we have

$$\frac{f(p_n, \bar{x} + sz) - f(p_n, \bar{x})}{s} \geq \frac{f(p_n, \bar{x} + \frac{z}{n}) - f(p_n, \bar{x})}{n^{-1}} \geq \frac{g(\bar{x} + \frac{z}{n}) - g(\bar{x})}{n^{-1}} \geq \langle z, v \rangle.$$

Letting n go to $+\infty$, we deduce that for all $s \in (0, 1)$,

$$\frac{f(\bar{p}, \bar{x} + sz) - f(\bar{p}, \bar{x})}{s} \geq \langle z, v \rangle.$$

Taking the limit when $s \rightarrow 0+$ and using the preceding lemma, we conclude that

$$Df_{\bar{p}}(\bar{x}, z) = \sup_{u \in \partial^- f_{\bar{p}}(\bar{x})} \langle z, u \rangle \geq \langle z, v \rangle,$$

which is a contradiction. \square

Before stating the next lemma, let us recall the definition of an exposed point.

Definition A.3. Let $f : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function. The set of exposed points of f is defined as the set of all $x \in \mathbb{R}^n$ such that there exists $x^* \in \partial^- f(x)$ such that

$$\forall x' \neq x, f(x') > f(x) + \langle x^*, x' - x \rangle.$$

Lemma A.4. Let $C \subset \mathbb{R}^m$ be a compact polyhedron and $f : C \rightarrow \mathbb{R}$ a convex Lipschitz function. If x is an extreme point of f and z in the tangent cone of C at x , then there exists a sequence x_n of exposed points of f with limit x such that

$$\vec{D}f(x_n; z) \rightarrow \vec{D}f(x; z).$$

Proof. Let us choose $y \in \partial^- f(x)$ such that $\vec{D}f(x; z) = \langle z, y \rangle$. We claim that x is an extreme point of $\partial^- f^*(y)$. Note at first that Fenchel's Lemma implies that $x \in \partial^- f^*(y)$. Assume then that x is not an extreme point of $\partial^- f^*(y)$, so that there exists a segment (x_1, x_2) containing x and included in $\partial^- f^*(y)$. It follows that $y \in \partial^- f(x')$ for all $x' \in (x_1, x_2)$, and thus that f restricted to this segment is affine, which contradicts the fact that x is an extreme point of f .

Using Theorem 25.6 of [30], there exists therefore a sequence y_n with limit y such that f^* is differentiable at y_n and the sequence $x_n := \nabla f^*(y_n)$ has limit x . (In the proof of Theorem 25.6 of [30], this statement is proved only for exposed points of $\partial^- f^*(y)$, but this extends easily to extreme points by a diagonal extraction). The sequence of points x_n is made of exposed points of f by Corollary 25.1.2 in [30] and

$$\vec{D}f(x_n; z) = \langle y_n, z \rangle \rightarrow \langle y, z \rangle = \vec{D}f(x; z).$$

□

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